Mathematical Reviews

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FOUNDATIONS

Ridder, J. Ueber den Aussagen- und den engeren Prädikatenkalkül. II. Nederl. Akad. Wetensch., Proc. **50**, 24–30 = Indagationes Math. **9**, 9–15 (1947).

The initial sections of this paper deal with the two axiom-sets previously developed for the propositional calculus [see part I, same Proc. **49**, 1153–1164 = Indagationes Math. **8**, 701–712 (1946); these Rev. **8**, 306]. By providing suitable arithmetical and set-theoretic models, the author proves that each set is consistent and that in one of the two sets the axioms are mutually independent. Next he states duality relations among the axioms of one set, and constructs a third set whose axioms are dual to the axioms of the second. Finally he shows that his enlarged version of the propositional algebra can be interpreted as an elementary probability calculus. In the remainder of the paper, the author extends these results to the restricted predicate calculus, two enlarged versions of which he constructs by adding new primitives, axioms and rules of inference to his two original axiom-sets. *G. D. W. Berry.*

van Dantzig, D. On the principles of intuitionistic and affirmative mathematics. Nederl. Akad. Wetensch., Proc. **50**, 918–929 = Indagationes Math. **9**, 429–440 (1947).

These pages, which are to be followed by additional sections and a bibliography, contain a general account of the intuitionistic position illustrated with numerous quotations from L. E. J. Brouwer. The work endeavors for "better understanding between intuitionists and 'ordinary mathematicians,'" and should, for that reason, be of interest to the general mathematical public. The author calls a statement stable in case it is intuitionistically equivalent to its double negation. Under a stable interpretation (insofar as such is possible), classical mathematics becomes a part of intuitionistic mathematics. The importance of intuitionism appears in the interpretation, stronger than the stable one, which it places upon statements. A stable interpretation of arithmetic has been given by Gödel [Ergeb. Math. Kolloq. **4**, 34–38 (1933)]. The author presents a stable interpretation of the Cantor definition of real numbers and of the integers regarded as a subclass of real numbers. A sample proof in stable form illustrates required variation from the usual mixture of interpretations found in classical proofs.
D. Nelson (Washington, D. C.).

Robinson, Raphael M. Primitive recursive functions. Bull. Amer. Math. Soc. **53**, 925–942 (1947).

This paper extends results of R. Péter [Math. Ann. **110**, 612–632 (1934); **111**, 42–60 (1935)]. A primitive recursive function is defined, starting with the zero function, the identity functions and the successor function, by $F(u, 0) = A(u)$; $F(u, Sx) = B(u, x, F(u, x))$ (u stands for $u_1, \cdots, u_n$). The recursion is called pure if $B(u, x, y)$ does not depend on x; it is called an iteration if B does not depend on u, and a pure iteration if B depends neither on x nor on u. Now every primitive recursive function can be obtained (1) by iteration with one parameter; (2) by recursion without parameter, if to the initial functions is adjoined either $|u-x|$ or $u+x$ and $Q(x)$; (3) by pure recursion with one parameter after adjunction of $P(x)$; (4) by pure iteration with one parameter after adjunction of $Q(x)$; (5) by pure recursion without parameter after adjunction either of $u+x$ and $E(x)$ or of $|u-x|$ and $Q(x)$. Here $P(0)=0$, $PSx=x$; $Q(x)=1$ for a square x, 0 for a non-square; $E(x)=x-[x^{\frac{1}{2}}]^2$. In (5) the adjunction of $u+x$ and $Q(x)$ is not sufficient. All recursive functions of one variable can be obtained by starting with $S(x)$ and $E(x)$, and repeatedly using $F(x)=A(x)+B(x)$, $F(x)=B(A(x))$, $F(x)=B^x(0)$, to construct a new function from known functions A and B.
A. Heyting (Amsterdam).

Goodstein, R. L. Transfinite ordinals in recursive number theory. J. Symbolic Logic **12**, 123–129 (1947).

The author generalizes representation of an integer n in a notation to the base k by digits less than k. The function $\phi_{k,\sigma}^{f(x)}(n)$ defines a representation of n to the base $\sigma > f(k)$ with digits $f(r)$, $0 \leqq r \leqq k$, for a suitably chosen $f(x)$. By means of this function and a set of symbols $\omega_0, \omega_1, \cdots$, expressions are defined recursively as follows: $\Omega_p(0, k) = k$, $\Omega_p(n+1, k) = \phi_{p(n), \omega_n}^{\Omega_p(n, x)}(k)$. An expression $\Omega_p(n, k)$ is called an ordinal of type n. A relation $R(\omega_0, \cdots, \omega_m)$ holds in case there exist a natural number c_0 and recursive functions $c_r(n_0, \cdots, n_{r-1})$ such that $R(n_0, \cdots, n_m)$ for $n_0 \geqq c_0$ and $n_{r+1} \geqq c_{r+1}(n_0, \cdots, n_r)$. It is shown that the decreasing ordinal theorem for ordinals of any type (the assertion that any decreasing sequence of ordinals terminates) is equivalent to an arithmetic statement. Pointing out the fact that $\Omega_p(n, k)$ is based on the operations of addition, multiplication and exponentiation for ordinal numbers, the author also discusses functions which define ordinal expressions of greater generality. In addition to the intrinsic interest of the work, it is of importance because of the role of a theory of ordinals in Gentzen's proof of the consistency of arithmetic [Math. Ann. **112**, 493–565 (1936)]. *D. Nelson.*

Markov, A. On certain insoluble problems concerning matrices. Doklady Akad. Nauk SSSR (N.S.) **57**, 539–542 (1947). (Russian)

Let a square matrix with integers as elements and determinant 1 be called "admissible." If $\{X_1, X_2, \cdots, X_p\}$ is a finite set of admissible n-matrices, the semi-group of all matrices expressible as products of X's is denoted by $S(X_1, X_2, \cdots, X_p)$. Theorem 1. If $n \geqq 4$ there is no algorithm (in the sense of Church-Kleene-Turing) for deciding whether, given two finite sets, $X_1, X_2, \cdots, X_p$ and $Y_1, Y_2, \cdots, Y_q$ of admissible n-matrices, the sets $S(X_1, X_2, \cdots, X_p)$ and $S(Y_1, Y_2, \cdots, Y_q)$ have a common member. Moreover (corollary) all the Y's, and all X's but one, can be chosen and fixed, so that the question remains undecidable when the remaining X varies; the number q can here be 2, and p can also be chosen independent of n.

Proof. By a theorem of E. L. Post [Bull. Amer. Math. Soc. 52, 264–268 (1946); these Rev. 7, 405] there is no algorithm for deciding whether, given p pairs, G_i, G_i' of words in two letters, a and b, there exists an identity $G_{i_1}G_{i_2}\cdots G_{i_m}=G'_{i_1}G'_{i_2}\cdots G'_{i_m}$; and Post's proof shows that all the G's and all but one of the G''s can be chosen so that the question is still undecidable when the remaining G' varies. The two matrices

$$A=\begin{pmatrix}1&1\\0&1\end{pmatrix},\quad B=\begin{pmatrix}1&0\\1&1\end{pmatrix}$$

generate a free semi-group $S(A,B)$ [J. Nielsen, Danske Vid. Selsk. Math.-Fys. Medd. 5, no. 12 (1924)]. Hence the question whether, given p pairs of matrices Z_i, Z_i', each equal to A or B, there exists a relation $\prod_1^m Z_{i_r}=\prod_1^m Z'_{i_r}$ is undecidable; or, what is the same, the question whether some $\prod(Z_{i_r}\dotplus Z'_{i_r})$ is in $S(A\dotplus A, B\dotplus B)$, where $\dotplus$ denotes the direct sum (4-matrix). This proves the case $n=4$, and the extension to $n>4$ is trivial.

Theorem 2. If $n\geqq 4$ there is no algorithm for deciding whether, given admissible n-matrices $X_1, X_2, \cdots, X_p$ and $Y_1, Y_2, \cdots, Y_q$, the set $S(X_1, X_2, \cdots, X_p)$ and the additive group $L(Y_1, Y_2, \cdots, Y_q)$ generated by the Y's, have a common member. All the Y's, and all the X's but one, can be fixed [corollary 1]; or all the X's can be fixed [corollary 2]; and the question is still undecidable when the remaining X, or the Y's, vary. (Here q can be 5, and p independent of n.) This is proved similarly, by taking Y_1, Y_2, Y_3, Y_4 to be the four 4-matrices $F_{rs}=E_{rs}+E_{sr}$, where E_{rs} has 1 in position (r,s) and 0 elsewhere. Then $L(Y_1, Y_2, Y_3, Y_4)$ consists of all matrices $Z\dotplus Z$ (Z admissible).

M. H. A. Newman (Manchester).

Szmielew, Wanda. On choices from finite sets. Fund. Math. 34, 75–80 (1947).

This paper gives another sufficient condition for the derivability of the statement $[Z]\rightarrow[n]$, which has been studied by Mostowski [Fund. Math. 33, 137–168 (1945); these Rev. 8, 3]. Here $[n]$ is the proposition that for every class K of sets each of which has n elements there is a function f_K defined for all X which are in K and such that f_K is in K, and $[Z]$ is the logical product of the r propositions $[n_1], \cdots, [n_r]$, Z being a finite nonempty class of positive integers and n being any positive integer. The condition given is that, in every decomposition of n into a sum of primes, at least one of the primes belongs to Z. The proof is by induction. A slightly stronger theorem is also proved.

R. M. Martin (Bryn Mawr, Pa.).

Beth, Evert W. Hundred years of symbolic logic. A retrospect on the occasion of the Boole-De Morgan centenary. Dialectica 1, 331–346 (1947).

Queiroz, Augusto. Infinite and criticism in mathematics. Anais Fac. Ci. Pôrto 26, 129–157 (1941). (Portuguese)

Walker, A. G. Durées et instants. Revue Sci. 85, 131–134 (1947).

Dans les théories physiques on affirme habituellement que l'expérience temporelle d'un observateur est un ensemble simplement ordonné d'instants et que l'instant est un concept de base. Mais le caractère temporel d'une expérience doit être décrit comme un intervalle et le concept d'instant doit être défini en termes d'intervalles. Dans cette note l'auteur donne un tel procédé. Soit D l'ensemble des durées (éléments indéfinis). Il est supposé que D est partiellement ordonné, c'est-à-dire qu'il y a une relation "$<$" qui peut exister entre deux durées. Si les durées a, b sont telles que à la fois $a<b$ et $a>b$ soient faux, on écrit $a|b$. Ces durées sont associées à des expériences qui ont lieu ensemble sans avoir nécessairement la même durée ($a|b$ n'implique pas $a=b$). Les relations $<$, $|$ et $>$ satisferont aux axiomes suivants: (1) $a|a$; (2) $a<b$, $b|c$, $c<d$ impliquent $a<d$. Alors un instant est définie comme un ensemble ordonné de trois classes (A, B, C) de durées telles que: (I) $A+B+C=D$; (II) A et B sont différents de zéro; (III) si $a\subset A$ et $b\subset B$, $a<b$; (IV) si $c<C$, il existe $a\subset A$ et $b\subset B$ tels que $a|c$ et $b|c$. L'auteur démontre que l'ensemble (I) des instants est simplement ordonné et fermé en ce sens que toute suite bornée monotone d'instants a une limite. Une durée c est dite contenir un instant $i(A,B,C)$ quand $c\subset C$. Il résulte que la classe d'instants contenus dans c est une intervalle de (I).

J. Haantjes (Amsterdam).

Altschul, Eugen, and Biser, Erwin. The validity of unique mathematical models in science. Philos. Sci. 15, 11–24 (1948).

***Franck, James. Remarks about the role of pure science in general education.** Studies and Essays Presented to R. Courant on his 60th Birthday, January 8, 1948, pp. 139–144. Interscience Publishers, Inc., New York, 1948. $5.50.

***Zwicky, F. The morphological method of analysis and construction.** Studies and Essays Presented to R. Courant on his 60th Birthday, January 8, 1948, pp. 461–470. Interscience Publishers, Inc., New York, 1948. $5.50.

ALGEBRA

***Schwerdtfeger, H. Symplectic groups and null systems.** Studies and Essays Presented to R. Courant on his 60th Birthday, January 8, 1948, pp. 371–382. Interscience Publishers, Inc., New York, 1948. $5.50.

Let P be a real nonsingular skew-symmetric matrix of order $2n$. The polarity in which the hyperplane associated with a point x is represented by the equation $x'Py=0$ is said to be a null system. The totality of nonsingular matrices S for which $S'PS=\sigma_0 P$ (σ_0 a real number) forms a group called by the author the symplectic group of the null system. He shows that for a given symplectic group there is only one null system with which it is associated. He also considers two null systems and their associated symplectic groups.

J. Williamson (Flushing, N. Y.).

Amato, Vincenzo. Sulla segnatura di un polinomio di matrice. Rend. Circ. Mat. Palermo 63, 113–120 (1942).

Results equivalent to but less concise than those of P. Muth [J. Reine Angew. Math. 125, 282–292 (1903)].

C. C. MacDuffee (Madison, Wis.).

Ferrari-Toniolo, A. Sul calcolo delle matrici applicato a quadripoli lineari semplificati e generalizzati. Univ. Roma e Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 1, 298–319 (1940).

In studying a network or any linear relationship $y_i=\sum_1^n a_{ik}x_k$ of square matrix A, we may desire to express any n out of the $2n$ variables $x_1, \cdots, y_n$ in terms of the others. The author treats only the case $n=2$, cataloging the various

possibilities, and discussing applications to combinations of 4-terminal networks. With the idea of facilitating the manipulation he introduces in addition to the inverse matrix giving the x_1, x_2 from the y_1, y_2, the "orthoverse" for y_1, x_2 from x_1, y_2; "metaverse" for y_1, x_1 from y_2, x_2; "paraverse" for x_2, y_2 from x_1, y_1; "roverse" ("riversa" = riga + inversa) for interchange of the two rows; similarly, "rometaroverse," etc.

The reviewer would note that the problem can be better handled, and generalized to any n, in terms of minors of the n by $2n$ matrix of the $2n$ variables, or by introducing the fundamental unit matrices E_j^i (all zeros except a 1 in the ith row, jth column). Then, if $i_1 \cdots i_n$ and $j_1 \cdots j_n$ are any two permutations of $1 \cdots n$, the matrix expressing the $x_{i_1}, \cdots, x_{i_r}, y_{j_{r+1}}, \cdots, y_{j_n}$ linearly in the remaining variables is (when the inverse exists)

$$\left(A\sum_1^r E_k^{i_k} - \sum_{r+1}^n E_k^{j_k}\right)^{-1}\left(\sum_1^r E_k^{j_k} - A\sum_{r+1}^n E_k^{i_k}\right),$$

which can be restated as a quite simple rule in actual computation. *L. C. Hutchinson* (Brooklyn, N. Y.).

Krasner, Marc. Théorie non-abélienne des corps de classes pour les extensions galoisiennes des corps de nombres algébriques: bimatrices; représentations bimatricielles des semi-groupes abéliens libres. C. R. Acad. Sci. Paris **225**, 785–787 (1947).

Krasner, Marc. Théorie non abélienne des corps de classes pour les extensions galoisiennes des corps de nombres algébriques: anneau des représentations d'un groupe; représentations associées du groupe de Galois et du semi-groupe des idéaux. C. R. Acad. Sci. Paris **225**, 973–975 (1947).

Let A and B be nonsingular square matrices with elements in the complex field, and let $A+B$ and $A\times B$ be their Kronecker sum and product, respectively. The ordered pair $\mathfrak{A}=(A, B)$ is called a bimatrix. Two bimatrices $\mathfrak{A}=(A, B)$ and $\mathfrak{A}'=(A', B')$ are equivalent if $A+B'$ and $B+A'$ are equivalent. The symbol 0 is used for the unique matrix of degree zero, so that $A+0=A$. The bimatrix $(A, 0)$ is identified with A and the "antimatrix" $(0, A)$ is denoted by $-A$. In general $-\mathfrak{A}=(B, A)$. Sums and (bimatric) products of bimatrices are defined by $\mathfrak{A}+\mathfrak{A}'=(A+A', B+B')$ and $\mathfrak{A}\times\mathfrak{A}'=(A\times A'+B\times B', A\times B'+B\times A')$. Under these operations the classes of equivalent bimatrices form a commutative ring. The author then defines bimatric representations of a free Abelian semigroup (that is, a semigroup S which contains a subset S_0 of prime elements such that every element of S is expressible uniquely as a product of nonnegative powers of prime elements). If $s\epsilon S$ and $R(s)$ and $R'(s)$ are bimatric representations of S then so also are $-R(s)$, $R(s)+R'(s)$ and $R(s)\times R'(s)$. With a suitable definition of equivalence the bimatric representations of S form a ring.

The matric product of the bimatrices $\mathfrak{A}$ and $\mathfrak{A}'$ is defined by $\mathfrak{A}\mathfrak{A}'=(AA', BB')$ where AA', BB' are ordinary matrix products, it being assumed here that A and A', and likewise B and B', have the same degree. If G is a group a bimatric representation of G is defined as a function Γ from G to the ring of bimatrices such that $\Gamma(\sigma\tau)=\Gamma(\sigma)\Gamma(\tau)$. Two bimatric representations $\Gamma(\sigma)=(A(\sigma), B(\sigma))$ and $\Gamma'(\sigma)=(A'(\sigma), B'(\sigma))$ are equivalent if there exists a nonsingular matrix C such that $A'(\sigma)+B(\sigma)=C[A(\sigma)+B'(\sigma)]C^{-1}$ for all σ in G. Since $-\Gamma$, $\Gamma+\Gamma'$ and $\Gamma\times\Gamma'$, all defined in the obvious way, are representations of G, it follows that the classes of equivalent representations form a commutative ring $\mathfrak{R}(G)$. A bimatric representation $\Gamma(\sigma)$ is called positive if it is equivalent to a matric representation, that is, $\Gamma(\sigma)\sim\Gamma'(\sigma)=(A(\sigma), 0)$. It is positively decomposable if $\Gamma\sim\Gamma'+\Gamma''$ where Γ' and Γ'' are positive. This is expressed by writing $\Gamma=\Gamma'(+)\Gamma''$. A subring $\mathfrak{R}$ of $\mathfrak{R}(G)$ is positively complete if it is generated by its positive elements and if $\Gamma\epsilon\overline{\mathfrak{R}}$ and $\Gamma=\Gamma'(+)\Gamma''$ implies $\Gamma'\epsilon\overline{\mathfrak{R}}$. It is proved that if $\overline{\mathfrak{R}}$ is a positively complete subring of $\mathfrak{R}(G)$ there exists a normal subgroup H of G such that $\overline{\mathfrak{R}}=\overline{\mathfrak{R}}(\bar{G})$, where $G=G/H$.

Now let k be an algebraic number field of finite degree and K a finite normal extension of k. Let $U_{K/k}$ be the free Abelian semigroup consisting of all integral ideals of k which are prime to the discriminant $D_{K/k}$ of K/k, $G_{K/k}$ the Galois group of K/k and $\langle K/k; \mathfrak{p}\rangle$ the Artin symbol of K/k for the prime ideal $\mathfrak{p}\epsilon U_{K/k}$. If $\Gamma(\sigma)$ is a bimatric representation of $G_{K/k}$ then the equivalence class of bimatrices to which $\Gamma(\sigma)$ belongs depends only on the class X_σ of conjugate elements of $G_{K/k}$ to which σ belongs and this class can be denoted by $\Gamma(X_\sigma)$. Then $R_\Gamma(\mathfrak{p})=\Gamma(\langle K/k; \mathfrak{p}\rangle)$ defines a bimatric representation R_Γ of the free Abelian semigroup $U_{K/k}$. Moreover $\Gamma\rightarrow R_\Gamma$ is an isomorphism of $\mathfrak{R}(G_{K/k})$ onto the ring of equivalence classes of bimatric representations of $U_{K/k}$. The character of a bimatric representation Γ and the Artin L-function of K/k for Γ are also discussed briefly.

D. C. Murdoch (Vancouver, B. C.).

Schmid, Hermann Ludwig. Zur algebraischen Theorie der Formen. I. Math. Ann. **120**, 1–9 (1947).

Suppose $x_1, \cdots, x_n$ are algebraically independent indeterminates over the field k. Let y_ν denote n forms of degree g_ν, $1\leqq\nu\leqq n$, with the resultant R taken with respect to the indeterminates x_ν. The author presents a proof of the following theorem of Hilbert. The elements x_ν are integrally dependent on the forms y_ν if and only if the vanishing of all y_ν implies the vanishing of all x_ν, or equivalently $R\neq 0$. Next it is shown that this theorem remains valid if positive weights are assigned to the x_ν and if the y_ν are homogeneous with respect to the weights. Then $[k(x):k(y)]=\prod_{\nu=1}^n g_\nu$ for $R\neq 0$. The proof of this generalization is first given for the special case $y_\nu=x_\nu^{g_\nu}$, $1\leqq\nu\leqq n$, then for indeterminate coefficients, and finally by specialization of the coefficients the desired formula for the degree is reached. The theorems are used for the proof of the following assertion. Let $t, b_2, \cdots, b_n$ be indeterminates over k, and $f(t)=t^n+b_2t^{n-2}+\cdots+b_n$ with $f'(t)=\prod_{\nu=1}^{n-1}(t-t_\nu)$. Set $g(\tau)=\prod_{\nu=1}^{n-1}(\tau-f(t_\nu))=\tau^{n-1}+a_1\tau^{n-2}+\cdots+a_{n-1}$. Then (i) $k(a_1, \cdots, a_{n-1})$ has the transcendence degree $n-1$ over k, (ii) the elements $b_2, \cdots, b_n$ are integrally dependent over $k[a_1, \cdots, a_{n-1}]$, and (iii) $[k(b_2, \cdots, b_n):k(a_1, \cdots, a_{n-1})]=n^{n-2}$, provided n is relatively prime to the characteristic of k.

O. F. G. Schilling (Chicago, Ill.).

THEORY OF GROUPS

Garrido, J. Sur la classification des formes cristallines. Anais Fac. Ci. Pôrto **30**, 22–44 (1945).

Die Gleichheit bzw. Verschiedenheit von Kristallpolyedern, kristallographischen Büscheln und kristallinen Konfigurationen wird auf Grund verschiedener zu Grunde gelegten Transformationsgruppen, die in eine bestimmte Rangfolge von Ober- und Untergruppen gebracht werden können, diskutiert. In der Natur herrschen gewisse Selek-

tionsprinzipien vor, welche aus der denkbaren Mannigfaltigkeit gewisse Typen als besonders ausgezeichnet erscheinen lassen. Im Gegensatz zur Ansicht des Autors hat man auch bei der Realisation der 219 Raumgruppen für bestimmte chemische Verbindungsklassen solche Auswahlregeln gefunden. *W. Nowacki* (Bern).

Piccard, Sophie. Sur les bases du groupe symétrique d'ordre 7!. C. R. Acad. Sci. Paris **225**, 1246–1247 (1947).

Piccard, Sophie. Sur les bases du groupe symétrique d'ordre 7!. C. R. Acad. Sci. Paris **226**, 42–43 (1948).

In her book on this subject [Mém. Univ. Neuchâtel **19** (1946); these Rev. **8**, 13] the author gives a table giving the possible choices of S and T such that $\{S, T\}=\mathfrak{S}_n$, where $n\leqq 6$. These notes extend this table to include $n=7$. *G. de B. Robinson* (Toronto, Ont.).

Amato, Vincenzo. I sottogruppi fondamentali del gruppo lineare secondo un modulo primo p. Rend. Circ. Mat. Palermo **62**, 81–104 (1939).

A subgroup H of a finite group G has been called "fundamental" by Cipolla [Rend. Accad. Sci. Fis. Mat. Napoli (3) **15**, 44–54 (1909)] if it is the normalizer of some element of G. The author determines the fundamental subgroups of the modular group $PSL(2, p)$ of order $\frac{1}{2}p(p^2-1)$ and tabulates their order, type, central and number of conjugate subgroups. Since $PSL(2, p)$ possesses, apart from the 1-element, only elements of orders $\frac{1}{2}(p-1)$, p, and $\frac{1}{2}(p+1)$ [Gierster, Math. Ann. **18**, 319–365 (1881)] and since their normalizers are well-known [cf., e.g., Burnside, Theory of Groups, 2d ed., Cambridge University Press, 1911, chap. 20], his results are not new. *K. A. Hirsch* (Leicester).

Sanov, I. N. On Burnside's problem. Doklady Akad. Nauk SSSR (N.S.) **57**, 759–761 (1947). (Russian)

Let $\mathfrak{G}_1$ be a group with k generators every element of which is of order n. Let $\mathfrak{G}$ be a free group of two generators. Let $D_{n\lambda}$ be that subgroup of $\mathfrak{G}$ generated by all the n^λ powers of the elements of $\mathfrak{G}$, where λ is a positive integer. If for all such λ the groups $\mathfrak{G}/D_{n\lambda}$ are finite, it is proved that $\mathfrak{G}_1$ is finite. Let $\mathfrak{H}$ be a free group of k generators, and let B_n be that subgroup of $\mathfrak{H}$ generated by the nth powers of the elements of $\mathfrak{H}$. Let $\phi_n(k)$ be the order of $\mathfrak{H}/B_n$. For particular values of n and k some lower bounds on this order are obtained. Typical are $\phi_6(2)\geqq 8\cdot 6^{25}$ and

$$\phi_4(k)\geqq \exp_2[1+k+2^k(k-1)].$$

F. Haimo (St. Louis, Mo.).

Sanov, I. N. A property of a representation of a free group. Doklady Akad. Nauk SSSR (N.S.) **57**, 657–659 (1947). (Russian)

Previous matric representations had been given for free groups, but these made use of transcendental numbers. Here it is shown that the group generated by

$$a=\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix},\qquad b=\begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}$$

is a free group. The proof depends on showing that no matrix $a^ib^ja^kb^m\cdots$ leaves the vector $v=[1, 0]$ invariant, where $i, j, k, m, \cdots$ are integers, positive or negative and all except possibly i different from zero. By direct calculation it is shown that in the vectors va^i, va^ib^j, $va^ib^ja^k$, $\cdots$ the numerical values of the greater argument are increasing. In a second theorem it is shown that the matrices of this free group are those unimodular matrices of the form $\begin{pmatrix}1+4m, & 2k\\ 2r, & 1+4h\end{pmatrix}$. *M. Hall, Jr.* (Columbus, Ohio).

Kiokemeister, Fred. A note on the Schmidt-Remak theorem. Bull. Amer. Math. Soc. **53**, 957–958 (1947).

Let $A_1, A_2, \cdots$ be a countable set of Ω groups, let G be the restricted direct product of the A_i, let Ω contain the inner automorphisms of G and let each A_i be a directly indecomposable group which satisfies both chain conditions for Ω subgroups. Then if G is also the restricted direct product of a second set of groups B_a the set of B's is countable and can be so ordered that A_i is operator isomorphic to B_i, $i=1, 2, \cdots$, and, moreover, for every j, G is the restricted direct product of the set $B_1, \cdots, B_j, A_{j+1}, A_{j+2}, \cdots$. The proof follows classical lines except for modifications of two auxiliary lemmas on properties of indecomposable Ω groups. *R. M. Thrall* (Ann Arbor, Mich.).

Faddeev, D. K. On factor-systems in Abelian groups with operators. Doklady Akad. Nauk SSSR (N.S.) **58**, 361–364 (1947). (Russian)

Functions are defined from the k-fold Cartesian product of a group $\mathfrak{F}$ into an additive Abelian group $\mathfrak{A}$ which has $\mathfrak{F}$ as a group of operators. Such functions are called k-factor sets (k-f.s.) if (1) holds (where the ϕ_i are in $\mathfrak{F}$):

$$(1)\quad a(\phi_2, \phi_3, \cdots, \phi_{k+1})+\sum_{i=1}^{k}(-1)^i a(\phi_1, \cdots, \phi_i\phi_{i+1}, \cdots, \phi_{k+1}) \\ +(-1)^{k+1}a(\phi_1, \phi_2, \cdots, \phi_k)\phi_{k+1}=0.$$

The 1-f.s. are the crossed characters and the 2-f.s. are the factor sets of Clifford and MacLane [Trans. Amer. Math. Soc. **50**, 345–406 (1941); these Rev. **3**, 194]. [See also MacLane and Schilling, ibid., 295–384; these Rev. **3**, 102.] Generalizations of so-called transformation sets, called principal k-f.s., are given by expressions as on the left of (1) with k replaced by $k-1$. The k-f.s. of $\mathfrak{F}$ in $\mathfrak{A}$ form a group $\mathfrak{N}^k_{\mathfrak{A}}$ in which the principal k-f.s. form a subgroup $\mathfrak{H}^k_{\mathfrak{A}}$. Their factor group $\mathfrak{M}^k_{\mathfrak{A}}$ is called the k-multiplicator group of $\mathfrak{F}$ in $\mathfrak{A}$. If $\mathfrak{F}$ is finite then the $\mathfrak{M}^k_{\mathfrak{A}}$ are torsion groups. If the order of $\mathfrak{F}$ divides every element of $\mathfrak{A}$ uniquely then $\mathfrak{H}^k_{\mathfrak{A}}=\mathfrak{N}^k_{\mathfrak{A}}$. Let $\mathfrak{F}$ be a group, and let $\mathfrak{b}$ be an additive Abelian group. Let $\mathfrak{F}$ be a group of operators for the group of functions $\mathfrak{F}^{\mathfrak{b}}$ under the rule $f^\phi(\psi)=f(\phi\psi)$. Then $\mathfrak{H}^k_{\mathfrak{F}^{\mathfrak{b}}}=\mathfrak{N}^k_{\mathfrak{F}^{\mathfrak{b}}}$. Every Abelian group $\mathfrak{A}$ either can be embedded in a group for which the $\mathfrak{H}$ equal the $\mathfrak{N}$, or it can extend an admissible subgroup of a group $\mathfrak{F}^{\mathfrak{b}}$. Let $\mathfrak{A}$ be an admissible subgroup of an additive Abelian group $\mathfrak{b}$. Denote the image of $\mathfrak{N}^{k-1}_{\mathfrak{b}}$ induced by the homomorphism $\mathfrak{b}\rightarrow\mathfrak{b}/\mathfrak{A}$ by $\overline{\mathfrak{N}}^{k-1}_{\mathfrak{b}}$. Then $(\mathfrak{N}^k_{\mathfrak{A}}\cap\mathfrak{H}^k_{\mathfrak{b}})/\mathfrak{H}^k_{\mathfrak{A}}\cong\mathfrak{N}^{k-1}_{\mathfrak{b}/\mathfrak{A}}/\overline{\mathfrak{N}}^{k-1}_{\mathfrak{b}}$. Moreover, if $\mathfrak{H}^k_{\mathfrak{b}}=\mathfrak{N}^k_{\mathfrak{b}}$, $\mathfrak{H}^{k-1}_{\mathfrak{b}}=\mathfrak{N}^{k-1}_{\mathfrak{b}}$, then $\mathfrak{M}^k_{\mathfrak{A}}\cong\mathfrak{M}^{k-1}_{\mathfrak{b}/\mathfrak{A}}$. Let E be the abstract unit group of a finite group $\mathfrak{F}$ [loc. cit.]. Let K be the additive group of reals modulo 1. The above results are used to show that $\mathfrak{M}^k_E\cong\mathfrak{M}^k_K$ for all positive integers k. This extends and strengthens a result of Clifford and MacLane for $k=1, 2$. *F. Haimo* (St. Louis, Mo.).

Vilenkin, N. On a class of complete orthonormal systems. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **11**, 363–400 (1947). (Russian. English summary)

Let G be a compact Abelian group. Then as is well known the members of the character group X of G form a complete system of orthonormal functions on G with respect to the Haar measure on the latter. Consequently they may be

used to expand functions on G in "Fourier series" whose terms are constant multiples of these characters. In this paper such expansions are considered in some detail for the special case in which G is zero dimensional and separable. In order to be able to discuss nonabsolute convergence the author introduces a sequential ordering in X. This ordering is related to the group structure of X but depends upon an infinite number of arbitrary choices and so is far from being intrinsic or natural in any sense. A related (nonsequential) ordering in G makes it possible to define functions of bounded variation. Using these notions the author obtains a number of theorems analogous to familiar ones in the classical theory of Fourier series. For example he shows that the Lebesgue constants of his series are $O(\log n)$ and that the series cannot converge to zero everywhere unless all the coefficients are zero. He asserts that by mapping G into the unit interval in a manner described by Freudenthal it is possible to deduce theorems about orthonormal sequences of functions of a real variable from his results.

G. W. Mackey (Cambridge, Mass.).

James, R. C., Michal, A. D., and Wyman, Max. Topological Abelian groups with ordered norms. Bull. Amer. Math. Soc. **53**, 770–774 (1947).

Normed Abelian groups in which every element has arbitrarily high nth roots, and with the norm taking values in an ordered Abelian group, are considered. The assumptions are adjusted so it can be (and is) proved that such a group is topological. Imbeddability in normed linear spaces is considered. The usual definition of differentials is given and shown to be related in the usual way to differentials on containing linear spaces.

W. Ambrose.

NUMBER THEORY

Min, Szu-hoa. On a system of congruences. J. London Math. Soc. **22**, 47–53 (1947).

Let p be a prime greater than 3, and consider the system of three congruences (mod p^α):

$$x_1+x_2+x_3\equiv y_1+y_2+y_3,$$
$$x_1^2+x_2^2+x_3^2\equiv y_1^2+y_2^2+y_3^2,$$
$$x_1^3+x_2^3+x_3^3\equiv y_1^3+y_2^3+y_3^3.$$

The author proves that the number of incongruent solutions of this system is, to within a factor $\{1+O(p^{-1})\}$, $p^{3\alpha+\alpha/3}$, $6p^{3\alpha+(\alpha-1)/3}$ or $16p^{3\alpha+(\alpha-2)/3}$ according as $p\equiv 0$, 1 or 2 (mod 3).

D. H. Lehmer (Berkeley, Calif.).

***Lind, Carl-Erik. Untersuchungen über die rationalen Punkte der ebenen kubischen Kurven vom Geschlecht Eins.** Thesis, University of Uppsala, 1940. 97 pp.

Working on a special type of rational plane cubic, the author rediscovers the transformation of order 2 of elliptic functions, and derives from it a criterion [substantially a special case of that given by A. Weil, Bull. Sci. Math. (2) **54**, 182–191 (1930)] for a rational point to arise from duplication in the group of rational points on the cubic. As usual [cf. loc. cit.], this leads to associated equations of the type $s^2=P(x, y)$, with P a biquadratic form. From this, using congruence properties and some special results of Euler, the author derives results about points of finite order and about the nonexistence of points of infinite order in certain cases, and solves completely a number of such cases.

A. Weil (Chicago, Ill.).

Venkataraman, C. S. Further applications of the identical equation to Ramanujan's sum $C_M(N)$ and Kronecker's function $\rho(M, N)$. J. Indian Math. Soc. (N.S.) **10**, 57–61 (1946).

The author derives some properties of the Ramanujan sum

$$C_M(N)=\sum_{h \bmod M}{}' \exp(2\pi ihN/M)$$

and of $\rho(N, M)$ ($=1$ if $(M, N)=1$ and 0 otherwise), using definitions (e.g., composition, cardinal component, principal function) and results of a preceding paper [Proc. Indian Acad. Sci. Sect. A. **24**, 518–529 (1946); these Rev. **8**, 445] and of Vaidyanathaswamy [Trans. Amer. Math. Soc. **33**, 579–662 (1931)]. [The reviewer remarks that these properties can also be formulated independently of those definitions and can be proved simply by changing summation orders.]

N. G. de Bruijn (Delft).

Venkataraman, C. S. The ordinal correspondence and certain classes of multiplicative functions of two arguments. J. Indian Math. Soc. (N.S.) **10**, 81–101 (1946).

On extending notions and results of the papers quoted in the preceding review a theory concerning "ordinal correspondence," "ordinal and modular functions of index r," "quasi-principal functions" (all relating to multiplicative functions $F(M, N)$) is developed here. The author seems to have overlooked the fact that all these notions become very simple, and all the theorems nearly trivial, when interpreted in terms of the generating functions

$$\Phi(x, y)=\sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\infty}F(p^\mu, p^\nu)x^\mu x^\nu, \qquad F(1, 1)=1,$$

of the prime components of F. For instance, the ordinal correspondence $F\rightarrow O(F)=f$ means $\Phi(x, y)\rightarrow\Phi(x, xy)=\varphi(x, y)$; modular functions of index r are generated by functions of the type $\Phi(x, x^r y)/(1-y)$, quasi-principal functions by $\Phi(x^r y)$, etc. Statement 7.61 (c) is false.

N. G. de Bruijn (Delft).

Lahiri, D. B. On Ramanujan's function $\tau(n)$ and the divisor function $\sigma_k(n)$. II. Bull. Calcutta Math. Soc. **39**, 33–52 (1947).

Ramanujan's function $\tau(n)$ is defined by

$$\Delta(x)=\sum_{n=1}^{\infty}\tau(n)x^n=x[(1-x)(1-x^2)(1-x^3)\cdots]^{24}.$$

No less than 171 congruences of the form

$$(1)\qquad A\tau(n)\equiv\sum_{k=0}^{6}P_k(n)\sigma_{2k+1}(n) \pmod{M}$$

are tabulated. Here A is a constant and P is a polynomial, both depending on n, while $\sigma_r(n)$ denotes the sum of the rth powers of the divisors of n. The moduli M are among the divisors of $2^{13}\cdot 3^6\cdot 5^3\cdot 7\cdot 11\cdot 13\cdot 691$. Not all these 171 congruences are independent since there are numerous congruences between the σ's. These were developed by the author in his previous paper I [same Bull. **38**, 193–206 (1936); these Rev. **8**, 567]. Also in many cases the constant A is not prime to the modulus M so that a congruence for $\tau(n)$ itself is not obtained modulo M. This occurs whenever M is divisible by 2^{11}, 3^6, 11 or 13. If one regards the functions $\sigma_r(n)$ as known and wishes only to express $\tau(n)$ in terms of them and various moduli the following five inde-

pendent congruences give the necessary information:

$$\begin{aligned}\tau(n)\equiv&(175n^2+499n)\sigma_7(n)\\&+(473n^3+235n^2-157n)\sigma_5(n)\\&+(224n^4-390n^3+155n^2+473n)\sigma_3(n)\\&+(48n^5+76n^4+396n^3-53n^2-105n)\sigma(n)\pmod{2^{10}},\\\tau(n)\equiv&\,n^2\sigma_7(n)-18n^2\sigma_5(n)+9n^2(2n+5)\sigma_3(n)-45n^3\sigma(n)\\&\hfill\pmod{3^5},\\\tau(n)\equiv&\,36n\sigma_9(n)-30n\sigma_5(n)-40n^2\sigma_3(n)+35n\sigma(n)\pmod{5^3},\\\tau(n)\equiv&\,n\sigma_3(n)\pmod{7},\\\tau(n)\equiv&\,\sigma_{11}(n)\pmod{691}.\end{aligned}$$

The author considers with Ramanujan the functions

$$\Phi_{r,s}(x)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}n^r m^s x^{mn}=\sum_{n=1}^{\infty}n^r\sigma_{s-r}(n)x^n$$

and finds all 41 products $P(x)$ of Φ's which can be expressed linearly in terms of other Φ's and the function $\Delta(x)$ in the form $MP(x)=A\Delta(x)+\sum A_{ks}\Phi_{k,s}(x)$, where M is an integer. Identifying coefficients of x^n on both sides and taking this identity modulo M one obtains a congruence relation of type (1). Further ones follow from this by using congruence relations between the σ_r's. *D. H. Lehmer.*

Bambah, R. P., and Chowla, S. Congruence properties of Ramanujan's function $\tau(n)$. Bull. Amer. Math. Soc. **53**, 950–955 (1947).

The authors prove two congruence properties of Ramanujan's function for the moduli 81 and 125 in the cases where n is prime to the modulus. Their results are

$$\tau(n)\equiv 5n^2\sigma_7(n)-4n\sigma_9(n)\pmod{125},$$
$$\tau(n)\equiv(n^2+k)\sigma_7(n)\pmod{81},$$

where $k=0$ or 9 according as $n\equiv 1$ or 2 (mod 3). Here $\sigma_k(n)$ denotes as usual the sum of the kth powers of all the divisors of n. The usual method of proof involving Ramanujan's functions P, Q and R is employed. *D. H. Lehmer.*

Hall, Marshall, Jr. On the sum and product of continued fractions. Ann. of Math. (2) **48**, 966–993 (1947).

It is proved that the sum $L(A)+L(B)$ (i.e., the set $\alpha+\beta$, $\alpha\varepsilon L(A)$, $\beta\varepsilon L(B)$) of two Cantor point sets obtained by subdividing the intervals A and B equals $A+B$ if the following conditions are both satisfied. (1) At any stage of the subdivision the length C_{12} of the interval C_{12}, deleted from an interval C, does not exceed the lengths C_1 and C_2 of the intervals retained; (2) the ratio ρ of the lengths of A and B satisfies $\frac{1}{3}\leqq\rho\leqq 3$. By applying this theorem it is proved that any real number can be represented in the form $\alpha+\beta$, and any number greater than or equal to 1 in the form $\gamma\delta$, where α, β, γ, δ are bounded regular continued fractions $[a_0, a_1, a_2, \cdots]$ whose partial quotients a_i satisfy $a_i\leqq 4$ $(i=1, 2, \cdots)$.

The second part of the paper contains considerations of a more arithmetical nature. A theory is developed concerning linear fractional forms $y=(ax-b)/(cx-d)$ (a, b, c, d are integers, $ad-bc=\pm N\neq 0$) connecting the continued fractions x and y. (The special case $N=1$ reduces to the well-known result that the partial quotients a_n and b_n satisfy $a_n=b_{n+k}$ for a certain integer k and for $n>n_0$). An algorithm is given which enables one to evaluate the sequence $\{b_n\}$ if $\{a_n\}$, a, b, c and d are given. Now the problem of representing a rational number p/q as the sum of two continued fractions x and y is related to the form $y=(-qx+p)/q$.

It is proved that among the representations of p/q as the sum of two bounded continued fractions at least one can be found where both fractions are periodic. The same holds for representations $p/q=\gamma\delta$. A special example shows that the inequalities $a_i\leqq 4$ cannot always be replaced by $a_i\leqq 2$. *N. G. de Bruijn* (Delft).

Kendall, D. G., and Rankin, R. A. On the number of Abelian groups of a given order. Quart. J. Math., Oxford Ser. **18**, 197–208 (1947).

It is known that the number $a(n)$ of Abelian groups of order $n=p_1^{\alpha}p_2^{\beta}\cdots p_l^{\lambda}$ equals $P(\alpha)P(\beta)\cdots P(\lambda)$, where $P(\alpha)$ is the number of partitions of α into positive parts. Using $\sum a(n)n^{-s}=\zeta(s)\zeta(2s)\zeta(3s)\cdots$ and a theorem of Landau [Nachr. Ges. Wiss. Göttingen. Math.-Phys. Kl. **1915**, 209–243], the authors prove that $\sum_{n\leqq x}a(n)=\alpha x-\beta x^{\frac12}+O(x^{\frac13}\log x)$, where $\alpha=\zeta(2)\zeta(3)\cdots$ and $\beta=-\zeta(\frac12)\zeta(\frac32)\zeta(\frac42)\zeta(\frac52)\cdots$.

It also is shown in an elementary manner that the sequence $a(1), a(2), \cdots$ possesses an asymptotic distribution function, a mean value, and that the first moment of the distribution function exists and equals the mean value. The distribution of $a(1), a(2), \cdots, a(1000)$ is computed and compared with the asymptotic distribution of the entire sequence. *P. Hartman* (Baltimore, Md.).

Rankin, R. A. On the closest packing of spheres in n dimensions. Ann. of Math. (2) **48**, 1062–1081 (1947).

The purpose of this paper is to extend a result of Blichfeldt's [Math. Ann. **101**, 605–608 (1929)]. Let C_n be an n-dimensional hypercube of edge L in n-dimensional Euclidean space; let $N(L)$ be the maximum number of hyperspheres of unit radius and content K_n which can be placed in C_n without overlapping each other or the sides of C_n. Then the packing constant ρ_n is defined to be $\rho_n=\lim_{L\to\infty}K_nN(L)/L^n$. The packing of the hyperspheres in C_n need not be regular, i.e., their centers need not form a lattice. The regular packing constant $\rho_n'=\lim_{L\to\infty}K_nN'(L)/L^n$, where $N'(L)$ is the maximum number of unit hyperspheres which can be placed in C_n without overlapping in any regular packing.

Proofs are given that both ρ_n and ρ_n' exist as unique limits; and that ρ_n' is related to the number γ_n which occurs in the theory of quadratic forms. A series of lemmas is developed and used to find expressions leading to the evaluation of bounds to the packing constant. The results are: $\rho_3\leqq 0.92998\cdots$, $\rho_3\leqq 0.82711\cdots$ (which is lower than Blichfeldt's two estimates) and $\rho_4\leqq 0.71197\cdots$. For $n=2$ the closest packing is known to be regular hexagonal, so that $\rho_2=\rho_2'=\pi/\sqrt{12}=0.90689968\cdots$. When $n>2$ it is not known whether the closest packing is regular, but it is shewn that $\rho_3\geqq\rho_3'=\pi/\sqrt{18}=0.74048\cdots$, and $\rho_4\geqq\rho_4'=\pi^2/16=0.61685\cdots$. *S. Melmore* (York).

Durrieu, Mauricio. A geometrical demonstration of the amount of empty space in a collection of spheres of equal diameter, arranged regularly, tangent in rows and layers in quincunxes. Ciencia y Tecnica **109**, 351–355 (1947). (Spanish. French summary)

Gál, István Sándor. Un théorème sur les approximations diophantines. C. R. Acad. Sci. Paris **225**, 844–846 (1947).

Define $<a, b>=(a, b)/[a, b]$, where (a, b) is the greatest common divisor and $[a, b]$ the least common multiple of a and b. Define

$$f(N)=\max_{n_i}\sum_{i,j\leqq N}<n_i, n_j>.$$

The author sketches a proof of the following result: $c_1N(\log\log N)^2 < f(N) < c_2N(\log\log N)^2$. *P. Erdős.*

Hinčin, A. Ya. On a limiting case of Kronecker's approximation theorem. Doklady Akad. Nauk SSSR (N.S.) **56**, 563–565 (1947). (Russian)

Theorem. If θ_i, α_i $(1\leqq i\leqq n)$ are real numbers, then the system of equations (1) $\theta_i x - y_i - \alpha_i = 0$ $(1\leqq i\leqq n)$ has integral solutions x, y_i if and only if there is a constant $\Gamma>0$ such that for any integers a_i, b, there is a further integer c satisfying

$$(2)\qquad \left|\sum_{i=1}^n a_i\alpha_i + c\right| < \Gamma\left|\sum_{i=1}^n a_i\theta_i + b\right|.$$

Proof. The condition is necessary since, by (1),

$$\sum_{i=1}^n a_i\alpha_i + c = x\left(\sum_{i=1}^n a_i\theta_i + b\right)$$

if $c = xb + \sum_{i=1}^n a_i y_i$. To show that it is sufficient, assume first that $n=1$, $\theta_1=\theta$ is irrational, $\alpha_1=\alpha$, and write θ_q, α_q for the shortest distance of θ and α from the nearest fractions of denominator $q\geqq 1$. Then, by hypothesis, $\alpha_q < \Gamma\theta_q$ for all integers $q\geqq 1$. Let p_n/q_n be the nth convergent of the continued fraction for θ, and let r_n be the integer with $\alpha_{q_n} = |\alpha - r_n/q_n|$. Put $x_n = r_n q_{n+1} - r_{n+1}q_n$, $y_n = r_n p_{n+1} - r_{n+1}p_n$. Then

$$|x_n| = q_nq_{n+1}\left|\frac{r_n}{q_n} - \frac{r_{n+1}}{q_{n+1}}\right| \leqq q_nq_{n+1}(\alpha_{q_n} + \alpha_{q_{n+1}}) \leqq \Gamma q_nq_{n+1}(\theta_{q_n} + \theta_{q_{n+1}}) = \Gamma,$$

since

$$\theta_{q_n} + \theta_{q_{n+1}} = \left|\frac{p_n}{q_n} - \frac{p_{n+1}}{q_{n+1}}\right| = \frac{1}{q_nq_{n+1}}.$$

Further,

$$|x_n\theta - y_n - \alpha| = \left|x_n\left(\theta - \frac{p_n}{q_n}\right) - \left(\alpha - \frac{r_n}{q_n}\right)\right| \leqq \Gamma\theta_{q_n} + \alpha_{q_n} \leqq 2\Gamma\theta_{q_n}.$$

Since x_n is bounded, x_n is a fixed integer x for an infinity of indices n, hence y_n is a fixed integer y, and so finally $x\theta - y - \alpha = 0$ since $2\Gamma\theta_{q_n}\to 0$. Next let $n\geqq 2$, and assume again that θ_1 is irrational. By (2), there exists for given integers p, q a third integer r such that $|q\alpha_1 - r| < \Gamma|q\theta_1 - p|$, and so $\alpha_1 = x\theta_1 - y_1$ with certain integral x, y_1, by what has just been proved. Choose, for $2\leqq i\leqq n$, integers k_i and l_i such that $k_i\theta_1 = l_i + \theta_i + \gamma_i$ with arbitrarily small $|\gamma_i|$. By (2), there exist integers m_i such that

$$|\delta_i| = |k_i\alpha_1 - m_i - \alpha_i| \leqq \Gamma|k_i\theta_1 - l_i - \theta_i| = \Gamma|\gamma_i|.$$

Now $\alpha_i = x\theta_i + (xl_i - y_1k_i - m_i) + (x\gamma_i - \delta_i)$. Here $xl_i - y_1k_i - m_i$ is an integer, and $|x\gamma_i - \delta_i|$ is arbitrarily small; hence the constant $\alpha_i - x\theta_i$ differs arbitrarily little from an integer, i.e., is itself an integer, $-y_i$ say, whence the assertion. The assertion is nearly obvious if all θ's are rational.

K. Mahler (Manchester).

Hinčin, A. Ya. On some general theorems of the theory of linear Diophantine approximations. Doklady Akad. Nauk SSSR (N.S.) **56**, 679–681 (1947). (Russian)

Let θ_{ij} $(1\leqq i\leqq s,\ 1\leqq j\leqq r)$ be rs real numbers, and let $L_j(x, y) = \sum_{i=1}^s \theta_{ij}x_i - y_j$ $(1\leqq j\leqq r)$. Assume that the system of inequalities (1) $L_j(x, y) = 0$ $(1\leqq j\leqq r)$, $\sum_{i=1}^s x_i^2 > 0$ has no solutions in integers x_i, y_j. The system of equations (2) $L_j(x, y) - \alpha_j = 0$ $(1\leqq j\leqq r)$ is called nondegenerate if it has no solution in integers. It is said to allow total approximate solutions of order $\beta>0$ if there is a $\gamma>0$ such that the inequalities

$$(3)\qquad |L_j(x, y) - \alpha_j| < 1/t,\quad |x_i| < \gamma t^\beta,\quad 1\leqq i\leqq s,\ 1\leqq j\leqq r$$

have integral solutions x_i, y_j for every $t\geqq 1$; it is said to allow partial approximate solutions of order β if, for some $\gamma>0$, (3) has integral solutions for an infinity of arbitrarily large t. Generalizing an earlier result of his [Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **11**, 105–110 (1947); these Rev. **9**, 10], the author proves the following theorem. There exist real numbers α_j for which (2) is nondegenerate and allows total approximate solutions of order β, if and only if (1) allows partial approximate solutions of order β.

The difficulty lies in the proof of sufficiency, which proceeds as follows. Choose an infinite sequence of positive λ_n with $\lambda_{n+1} > 2\lambda_n$ such that the inequalities

$$\xi_j^{(n)} = |L_j(u^{(n)}, -v^{(n)})| < \lambda_n^{-1},\quad |u_i^{(n)}| < \gamma\lambda_n^\beta,\quad \sum_{i=1}^s u_i^{(n)^2} > 0$$

have integral solutions u_i, v_j, but that the stronger inequalities

$$|L_j(x, y)| < 2/\lambda_{n+1},\quad |x_i| < 2\gamma'\lambda_n^\beta,\quad \sum_{i=1}^s x_i^2 > 0,$$

where $\gamma' = \gamma\sum_0^\infty 2^{-n\beta}$, have no solution in integers. Put

$$\alpha_j = \sum_{n=1}^\infty \xi_j^{(n)},\quad k_i^{(n)} = \sum_{q=1}^{n-1} u_i^{(q)},\quad l_j^{(n)} = \sum_{q=1}^{n-1} v_j^{(q)}.$$

Then the series α_j evidently converge, and

$$|k_i^{(n)}| < \gamma\sum_{q=1}^{n-1}\lambda_q^\beta < \gamma'\lambda_{n-1}^\beta,$$

$$R_j^{(n)} = |L_j(k^{(n)}, -l^{(n)}) - \alpha_j| \leqq \sum_{q=1}^{n-1}|\xi_j^{(q)} - \alpha_j| \leqq \sum_{q=n}^\infty |\xi_j^{(q)}| < 2/\lambda_n.$$

Determine now n, as a function of $t\geqq\lambda_1$, by $\lambda_{n-1}\leqq 2t < \lambda_n$, so that

$$R^{(n)} < 1/t,\quad |k_i^{(n)}| < \gamma'\lambda_{n-1}^\beta \leqq 2^\beta\gamma' t^\beta,$$

as asserted. The equations (2) are nondegenerate, since if $\alpha_j = L_j(a, -b)$, then

$$R_j^{(n)} = |L_j(k^{(n)} - a, b - l^{(n)})| < 2/\lambda_n,\quad |k_i^{(n)} - a_i| < 2\gamma'\lambda_{n-1}^\beta,$$

so that finally $k_i^{(n)} = a_i$, $u_i^{(n)} = k_i^{(n+1)} - k_i^{(n)} = 0$, contrary to hypothesis. *K. Mahler* (Manchester).

Rogers, C. A. A note on a problem of Mahler. Proc. Roy. Soc. London. Ser. A. **191**, 503–517 (1947).

This paper answers the following problem proposed by Mahler [same Proc. Ser. A. **187**, 151–187 (1946), p. 167; these Rev. **8**, 195]: does every critical lattice of a star body $F(x)\leqq 1$, for given $\epsilon>0$, contain n linearly independent lattice points P for all of which $1\leqq F(P) < 1+\epsilon$? The author shows that the star body $\min\{|x^2-xy-y^2|,\ |y^2-xy-x^2|\}\leqq 1$ possesses the single critical lattice of all points with integral coordinates and that its only admissible lattices Λ with $|\Delta(\Lambda)| < 1.6$ are obtained by magnifying the critical lattice. The proofs depend on some elementary results from the theory of minima of indefinite binary quadratic forms. Star bodies are then constructed by adding certain narrow rectangular strips, centered on the x-axis, to the above star body. The single critical lattice of each of these bodies is shown to be a magnification of the lattice of points of integral coordinates and has the property that its only points in the star body are the origin and the two points on the x-axis nearest to the origin. Thus Mahler's problem is

answered in the negative for the 2-dimensional case. An induction proof of some length (seven pages) is given showing that star bodies with similar properties exist for all dimensions. Examples are given illustrating the lack of continuity of $\Delta(K)$ for certain variable star bodies K.

D. Derry (Vancouver, B. C.).

Rogers, C. A. A note on irreducible star bodies. Nederl. Akad. Wetensch., Proc. **50**, 868–872 = Indagationes Math. **9**, 379–383 (1947).

The principal results of a paper of Mahler [same Proc. **49**, 331–343 = Indagationes Math. **8**, 200–212 (1946); these Rev. **8**, 12] are obtained by a similar but simpler treatment. This simplification is effected by introducing the notion of an irreducible point, a point P on the boundary of a star body H with the property that, for every star body K not containing P with $K<H$, $\Delta(K)<\Delta(H)$. The results are easy consequences of the following theorem and its converse. For every irreducible point P a lattice Λ can always be constructed which is H-admissible except for lattice points in given neighborhoods of P and $-P$ and with $d(\Lambda)<\Delta(H)$.

D. Derry (Vancouver, B. C.).

Ren'i, A. A. On some new applications of the method of Academician I. M. Vinogradov. Doklady Akad. Nauk SSSR (N.S.) **56**, 675–678 (1947). (Russian)

Let $\chi(n)$ be a nonprincipal primitive character mod D; let α and β be irrationals whose continued fractions are $\alpha=1/a_1+1/a_2+\cdots$, $\beta=1/b_1+1/b_2+\cdots$, where

$$a_n\leqq(\log D)^t n^\rho, \quad b_n\leqq(\log D)^t n^\rho,$$

$t>0$ and $\rho\geqq0$ being constants. The author proves

$$\sum_{\alpha D<n<\beta D}\chi(n)=O(1+t+\rho)(\log\log D)^2D^{\frac{1}{2}}.$$

If $\chi(-1)=1$, the result is also true for $\alpha=0$. The proof proceeds as follows. It follows from the classical theory of L-series that it suffices to prove

$$\sum_{n=1}^{D}\bar{\chi}(n)n^{-1}e^{2\pi i n\alpha}=O(1+t+\rho)(\log\log D)^2.$$

The author writes $n=pm$, where p is the greatest prime dividing n. If the summation is only extended over those values of n for which either $p\leqq(\log D)^{5+t+\rho}$ or $m\leqq(\log D)^{5+t+\rho}$, the result is straightforward. For the other values of n, the author uses Vinogradov's argument

$$\Big|\sum_n\Big|^2=\Big|\sum_p\sum_m\Big|^2\leqq\sum_p p^{-2}\sum_p\Big|\sum_m\chi(m)m^{-1}e^{2\pi i p m\alpha}\Big|^2,$$

from which the result follows by a routine calculation.

H. Heilbronn (Bristol).

Linnik, Yu. V. On the expression of L-series by means of ζ-functions. Doklady Akad. Nauk SSSR (N.S.) **57**, 435–437 (1947). (Russian)

The author proves the identity

$$\sum_{n=1}^{\infty}\chi(n)a_nn^{-s}e^{-n/N}=\epsilon(\chi)D^{-\frac{1}{2}}(2\pi i)^{-1}$$
$$\times\int A(w+s)\Gamma(w)\sum_{m=0}^{D-1}\bar{\chi}(-m)(N^{-1}+2\pi imD^{-1})^{-w}dw,$$

where $\chi(n)$ is a primitive character mod D, $|\epsilon(\chi)|=1$, $N>0$ and the integral is extended over a vertical line in the half-plane of convergence of the Dirichlet series $A(s)=\sum a_nn^{-s}$. The proof is based on Mellin's formula and the fact that the residue classes n (mod D) form a multiplicative group with characters $\chi(n)$ and an additive group with characters $e^{2\pi imn/D}$. As an example the author gives the formula

$$\sum_{n=2}^{\infty}\chi(n)\Lambda(n)e^{-n/N}=-\epsilon(\chi)D^{-\frac{1}{2}}\sum_\rho\Gamma(\rho)$$
$$\times\sum_{m=0}^{D-1}\bar{\chi}(-m)(N^{-1}+2\pi imD^{-1})^{-\rho}+O(\log^2 N),$$

where ρ runs through those nontrivial zeros of $\zeta(s)$ whose imaginary parts are in absolute value less than $N\log^2 N$. Finally, the author states without proof that if $\xi(s)\neq0$ for $\sigma>1-\eta$, $0\leqq\eta<\frac{1}{4}$, $\delta>0$, then

$$\sum_p e^{2\pi ip^{1/m}}e^{-(p/N)^{1/m}}=O(N^{1-1/(2m)-\eta/m+\delta}).$$

H. Heilbronn (Bristol).

***van der Blij, Frederik. Theta functions of degree m.** Thesis, University of Leiden, 1947. 47 pp.

The theta functions considered are of the form

$$\theta_{GH}(Z\,|\,T;P,\nu)=\sum_{M\equiv P(\text{mod }\Delta\nu)}e\left\{\frac{(M-P)H'}{\Delta\nu}\right\}$$
$$\times e\left\{\frac{(M+\frac{1}{2}G)T(M+\frac{1}{2}G)'}{\Delta\nu}\right\}e\{2Z(M+\tfrac{1}{2}G)'\},$$

where $e\{X\}=e^{(\pi i/\Delta)\sigma(QX)}$; G, H, P, Q are integral matrices satisfying certain conditions; Z and T are certain matrices with complex elements; ν is a positive integer; $\Delta=|Q|$; $\sigma(X)$ is the trace of X. These functions are formally similar to the theta functions considered by H. D. Kloosterman [Ann. of Math. (2) **47**, 317–375, 376–447 (1946); these Rev. **9**, 12, 13] but vectors have here been replaced by matrices and it has become necessary to introduce the trace into the exponentials; cf. C. L. Siegel [Ann. of Math. (2) **36**, 527–606 (1935)]. The results closely parallel some of those of Kloosterman. The dependence of the theta functions on the parameters is determined. Also studied is the behavior of the functions under generalized modular transformations. Here Z is replaced by $Z(CT+D)^{-1}$ and T by $(AT+B)(CT+D)^{-1}$ with A, B, C, D matrices such that $U=\left(\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right)$ is modular; i.e., $U'IU=I$ with $I=\left(\begin{smallmatrix}N & E\\ -E & N\end{smallmatrix}\right)$, N the zero matrix, E the unit matrix; cf. Siegel [Math. Ann. **116**, 617–657 (1939); these Rev. **1**, 203]. Similarly, generalized Gaussian sums are defined and analogous transformation formulas for them are determined.

H. S. Zuckerman (Seattle, Wash.).

van der Blij, F. A matric representation of binary modular congruence groups of degree m. Nederl. Akad. Wetensch., Proc. **50**, 942–951 = Indagationes Math. **9**, 453–462 (1947).

This first communication announces results whose proofs will apparently be completed in further publications. Continuing the work reviewed above the author states that he will obtain generalizations of more of Kloosterman's results and will finally obtain matrix representations of certain modular congruence groups. Some preliminary results are obtained in this communication. *H. S. Zuckerman.*

van der Blij, F. A matric representation of binary modular congruence groups of degree m. II. Nederl. Akad. Wetensch., Proc. **50**, 1084–1091 = Indagationes Math. **9**, 498–505 (1947).

Continuing the work reviewed above the author obtains generalizations of more of Kloosterman's results. The final results are matrix representations of certain modular congruence groups. *H. S. Zuckerman* (Seattle, Wash.)

ANALYSIS

Belz, Maurice H. Note on the Liapounoff inequality for absolute moments. Ann. Math. Statistics **18**, 604–605 (1947).

Let $F(x)$ be a distribution function and $\nu_r=\int_{-\infty}^{\infty}|x|^r dF(x)$. The author gives a short proof (using differentiation with respect to r) of the inequality $\nu_b^{a-c}\leqq\nu_c^{a-b}\nu_a^{b-c}$, where $c\leqq b\leqq a$.

R. P. Boas, Jr. (Providence, R. I.).

Ilieff, Ljubomir. Über ein Problem von D. Pompeiu. Annuaire [Godišnik] Univ. Sofia. Fac. Phys.-Math. Livre 1. **42**, 83–96 (1946). (Bulgarian. German summary)

Let D be a sector of a circle, or of a concentric circular annulus, whose central angle is an irrational multiple of π, and let $F(\xi,\eta)$ be continuous for all real ξ,η. Generalizing a problem of Pompeiu, the author investigates the continuous solutions $f(x,y)$ of the identity

$$\iint_D f(\xi+x\cos\alpha-y\sin\alpha,\ \eta+x\sin\alpha+y\cos\alpha)\,dxdy\equiv F(\xi,\eta).$$

It is shown that there is no continuous solution if $F(\xi,\eta)\not\equiv$ constant, and that the only continuous solution is a constant if $F(\xi,\eta)\equiv$constant. A more general problem is proposed.

E. F. Beckenbach (Los Angeles, Calif.).

Fuchs, W. H. J. On a theorem of Mandelbrojt. J. London Math. Soc. **22**, 19–25 (1947).

The author proves the following theorem. Hypotheses: (1) $0<\lambda_1<\lambda_2<\cdots$; (2) on setting $\psi(r)=\exp\left(2\sum_{\lambda_\nu<r}\lambda_\nu^{-1}\right)$ ($\sum=0$ for $r<\lambda_1$), $S(r)=\text{l.u.b.}_{0\leqq x\leqq r}([\psi(r)]^x/v(x))$, where $v(x)\geqq 1$, $T^*(r)=\text{l.u.b.}_{q\geqq 1}r^q/(M_q v(q))$, suppose

$$\int^{\infty}\log S(r)r^{-2}dr<\infty,\quad \int^{\infty}\log T^*(r)r^{-1-1/(2a)}dr=\infty;$$

(3) $F(s)=F(\sigma+it)$ is regular and bounded in $|t|<a\pi$, and for $n\geqq 1$: $|F(s)-\sum_{\nu=1}^n d_\nu e^{-\lambda_\nu s}|<M_q e^{-q\sigma}$ ($|t|<a\pi$). Then $F(s)\equiv 0$; $d_n=0$ ($n=1,2,\cdots$). This allows the author to give a new proof and to generalize a simple case of a theorem of the reviewer [reference in the following review]. [The reviewer has since proved more general theorems than those quoted by the author; see the following review.]

To prove his theorem the author proves the following lemma. If $\int^\infty\log S(r)r^{-2}dr<\infty$, then there is a function $f(u)$ such that

$$0<\int_0^\infty|f(u)|u^x du<v(x)\qquad (x>0),$$

$$\int_0^\infty|f(u)|u^{-\frac{1}{2}}du<K<\infty,\quad \int_0^\infty f(u)u^{\lambda_n}du=0$$

($n=1,2,\cdots$). [It may be remarked that this lemma furnishes a converse theorem to the reviewer's theorems on functions which are necessarily identically zero if "many" of their derivatives are zero at the origin; cf. the following review; Fourier's transform applied to $u^{-\frac{1}{2}}f(u)$ furnishes such a converse theorem.]

S. Mandelbrojt (Paris).

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Mandelbrojt, S. Une inégalité fondamentale. Ann. Sci. École Norm. Sup. (3) [illegible], 351–378 (1947).

The author's fundamental theorem, a generalization of his earlier results [Trans. Amer. Math. Soc. **55**, 96–131 (1944); these Rev. **5**, 176], deals with inequalities for the coefficients of a Dirichlet series, not in general convergent, which represents a given analytic function in a specified sense in a suitable domain. The theorem is stated in several variant forms, of greater or less generality; only one of them will be quoted here. Let the nondecreasing positive sequence $\{\lambda_n\}$ have finite upper density $D^\cdot=\limsup\lambda^{-1}N(\lambda)$, where $N(\lambda)$ is the number of λ_n not exceeding λ; define $D^\cdot(\lambda)=\sup_{x\geqq\lambda}x^{-1}N(x)$. Let Δ be a domain in the $s=\sigma+it$ plane defined by $\sigma>a_0$, $|t|<\pi g(\sigma)$, $g(\sigma)$ continuous, of bounded variation and such that $g(\sigma)>D^\cdot$ and $\lim g(\sigma)>D^\cdot$. Let $F(s)$ be analytic in Δ and continuable from Δ into the circle $|s-s_0|\leqq\pi R$ along a channel of width greater than $2\pi D^\cdot$; a channel is a sum of circles of constant radius whose centers describe a Jordan arc. Let the sums $\sum_1^m d_k e^{-\lambda_k s}$, $m>n$, represent $F(s)$ in Δ with "logarithmic precision" $p_n(\sigma)$ such that

$$\text{(A)}\qquad \int^\infty p_n(\sigma)\exp\left\{-\tfrac{1}{2}\int^\sigma\frac{du}{g(u)-D^\cdot(p_n(u))}\right\}d\sigma=\infty;$$

that is, let

$$\inf_{m\geqq n}\sup_{\sigma\geqq x}\left|F(s)-\sum_{k=1}^m d_k e^{-\lambda_k s}\right|\leqq e^{-p_n(x)}$$

in Δ, where $p_n(x)$ increases to ∞ and satisfies (A). Then

$$|d_n|\leqq\tfrac{1}{2}\pi^2\lambda_n R e^{2\nu(R)}\Lambda_n^*\max_{|s-s_0|\leqq\pi R}|F(s)|e^{\lambda_n\Re(s_0)},$$

where

$$\nu(D)=\sup_{\lambda>0}\int_0^\lambda(x^{-1}N(x)-D)dx$$

and

$$\Lambda_n^*=\prod_{m\neq n}\lambda_m^2/|\lambda_n^2-\lambda_m^2|;$$

estimates for Λ_n^* in terms of λ_n are given in the paper.

Applications are indicated to the analytic continuation of functions represented by Dirichlet series, generalizing those previously given [loc. cit.]. More details are given of applications to generalized quasi-analyticity. The following theorem, one of several, will serve as an example. Let $f(x)$ be of class C^∞ in $0\leqq x<\infty$ and bounded there, together with all its derivatives. Let $\{\nu_n\}$ be a sequence of nonnegative integers, $\nu_1=0$; $\{q_n\}$ is the complementary sequence, with $D^\cdot<\frac{1}{2}$; and $f^{(\nu_n)}(0)=0$. Let $C_f(\sigma)=\sup_{n\geqq 1}(n\sigma-\log m_n)$, where $m_n=\sup_{x\geqq 0}|f^{(n)}(x)|$, and suppose that

$$\int^\infty C_f(\sigma)\exp\left\{-\int^\sigma\frac{du}{1-2D^\cdot(C_f(u))}\right\}=\infty.$$

Then $f(x)\equiv 0$. Such results are proved by applying the fundamental theorem to

$$F_a(s)=\int_0^\infty\exp(-xe^{s-a})f(x)dx,\qquad a>0.$$

Finally, the author proves the following result in the opposite direction, to show that his results cannot be appreciably sharpened. With ν_n and q_n as in the last theorem, let $\log q_n/n\to 0$. There exists an $f(x)$ of C^∞ in $0\leqq x<\infty$, bounded there together with all its derivatives, and with $f^{(\nu_n)}(0)=0$ ($n\geqq 1$), such that for sufficiently large σ

$$C_f(\sigma)\geqq C\,\mathrm{cnv}\,N(e^{-\sigma})+\int_0^\sigma N(e^u)du,$$

where C is any given positive constant less than log 2; here "cnv" indicates the following operation: given a function

$R(\sigma)$, bounded below, cnv $R(\sigma)$ is the largest convex function not exceeding $R(\sigma)$.

[The author wishes to correct the following typographical errors: p. 355, l. 7, read $-3\log(\underline{h_n D(\sqrt{2}\lambda_n))}$ instead of $-3\log(\lambda_n D(\sqrt{2}\lambda_n))$; p. 362, l. 7, replace $\overline{\text{borne}}\ P_m(\sigma)$ by $\underline{\text{borne}}\ P_m(\sigma)$.] *R. P. Boas, Jr.* (Providence, R. I.).

Mandelbrojt, Szolem, et Wiener, Norbert. Sur les fonctions indéfiniment dérivables sur une demi-droite. C. R. Acad. Sci. Paris **225**, 978–980 (1947).

Let $f(x)$ belong to C^∞ on $0\leqq x<\infty$ and let $|f^{(n)}(x)|\leqq m_n$. The authors outline a proof of the following theorem on generalized quasi-analyticity. Let $f^{(2n)}(0)=0$, $f^{(2p_n+1)}(0)=0$. Let

$$P(z)=\prod_{n=1}^{\infty}\frac{2p_n+z}{2p_n-z}e^{-z/p_n}$$

and let $\mathfrak{m}(x)$ be defined by putting $\log\mathfrak{m}(x)$ equal to the ordinate of the Newton polygon of the points $(n, \log m_n)$ if $\lim\log m_n/n=\infty$, $\log\mathfrak{m}(c)=c$ if $\liminf\log m_n/n=c<\infty$; let $\mathfrak{m}_n=\mathfrak{m}(n)$. If there exist an increasing $N(\sigma)$ and a $G(\sigma)$ of bounded variation such that $0<l<G(\sigma)<\frac{1}{2}\pi$, $\mathfrak{m}_{2n+1}|P(2n+1)|=O(1)$ and

$$\log\{\mathfrak{m}[e^\sigma\cos G(\sigma)]|P(e^{\sigma+iG(\sigma)})|\}<-N(\sigma),$$

$$\int^\infty N(\sigma)\exp\left\{-\tfrac{1}{2}\pi\int^\sigma[G(u)]^{-1}du\right\}d\sigma=\infty,$$

then $f(z)\equiv 0$. The proof depends on defining

$$H(\omega)=\underset{B\to\infty}{\text{l.i.m.}}\int_{-B}^{B}t^{-1}f(t)e^{it\omega}dt,$$

so that

$$\int_0^\infty|H(\omega)|^2\omega^{2\nu}d\omega\leqq C^\nu\mathfrak{m}_{\nu+1}^2,$$

and considering

$$F(z)=\int_0^\infty H(\omega)e^{z\log\omega}d\omega,$$

which is analytic for $x>0$ and has $F(2p_n)=0$. A theorem of Mandelbrojt and MacLane [Trans. Amer. Math. Soc. 61, 454–467 (1947); these Rev. 8, 508] can be applied to $F(z)P(z)(1+z)^{-q}e^{-Bz}$ with suitable q and B to infer that $F(z)\equiv 0$ and so $f(z)\equiv 0$. *R. P. Boas, Jr.*

Mandelbrojt, Szolem, et Wiener, Norbert. Quasi-analyticité générale et théorèmes du type Phragmén-Lindelöf. C. R. Acad. Sci. Paris **226**, 47–49 (1948).

With the notations of the preceding review, the authors show that, if $f(z)\not\equiv 0$ and $f^{(2n)}(0)=f^{(2p_n+1)}(0)=0$, and if $A(z)$ is meromorphic in $\Re(z)>0$, with simple poles at $2p_n$, and bounded in every region $0<\alpha\leqq\Re(z)\leqq\beta$, $|z-2p_n|\geqq\epsilon>0$, then

$$\liminf_{x\to\infty}x^{-1}\log\left\{[\mathfrak{m}(x+2)]^2\int_{-\infty}^{\infty}|A(x+iy)|^2dy\right\}>-\infty.$$

The theorem allows one to show that $f(x)\equiv 0$ if suitable relations exist between m_n and p_n; and conversely, using inverse results of Mandelbrojt [see the second preceding review], it gives conditions for the existence of a meromorphic function with given poles and given behavior on vertical lines. *R. P. Boas, Jr.* (Providence, R. I.).

Theory of Sets, Theory of Functions of Real Variables

***Eyraud, Henri. Leçons sur la Théorie des Ensembles, les Nombres Transfinis et le Problème du Continu.** Institut de Mathématiques, Lyon, 1947. 65 pp.

Notes of a course of lectures, developing the elementary theory of transfinite numbers in a semi-formal manner, concluding with another [closely related to or identical with that summarized in C. R. Acad. Sci. Paris **224**, 85–87 (1947); these Rev. **8**, 448, and distinct from that reviewed in these Rev. **7**, 512] attempted proof of the continuum hypothesis. The author's proof breaks down in the fifth line from the foot of page 50, where he makes implicit use of the inequality $\aleph_1\cdot 2^{\aleph_0}<\aleph_2$, which itself implies the continuum hypothesis. *J. W. Tukey* (Princeton, N. J.).

Obreanu, F. La puissance de certaines classes de fonctions. Duke Math. J. **14**, 377–380 (1947).

The class of one-to-one correspondences of an infinite E with itself has the same power as the class of all functions from E to E. The proof uses the axiom of choice and in particular the fact that the power of E is an aleph. The class of real functions continuous in the sense of Darboux (i.e., taking all intermediate values) has the same power as the class of all real functions. *J. W. Tukey.*

Rohlin, V. On the classification of measurable decompositions. Doklady Akad. Nauk SSSR (N.S.) **58**, 29–32 (1947). (Russian)

Rohlin, V. On the problem of the classification of automorphisms of Lebesgue spaces. Doklady Akad. Nauk SSSR (N.S.) **58**, 189–191 (1947). (Russian)

The author considers complete measure spaces M with measures μ such that $\mu(M)=1$. If an object (e.g., a set, a function or a decomposition) is associated with each of two families of measure spaces, then the objects P and Q are "isomorphic" if there is a one-to-one, measurability and measure preserving transformation between corresponding spaces in the two families which (modulo sets of measure zero) carries P into Q. The "isomorphism type" of an object P, denoted by $\tau(P)$, is the class of all objects isomorphic to P.

A measure space M is a "Lebesgue space" if it contains a sequence $S=\{D_n\}$ of measurable sets such that (a) the completion of the Borel field generated by S is the class of all measurable sets, (b) every two distinct points of M may be separated by sets of S, and (c) if, for each $n=1, 2, \cdots$, $E_n=D_n$ or else $E_n=M-D_n$, then $\cap_n E_n\neq 0$. If the (at most countably many) points of positive measure in a measure space M are arranged in a sequence so that their measures are nonincreasing, the measure of the nth such point is denoted by $m_n(M)$. A decomposition ζ of a Lebesgue space is "measurable" if there exists a countable class of ζ-sets (i.e., sets which are unions of sets of the decomposition) by means of which any two sets of the decomposition may be separated.

The "factor space" M/ζ of a Lebesgue space by a measurable decomposition may be made into a (Lebesgue) measure space in a natural way, and a measure μ_C may be defined on each set C of the decomposition so that the measure of every measurable subset of M is obtained by integration (over M/ζ) of the μ_C-measure of its intersection with C. If U is an automorphism of a Lebesgue space M (i.e., an isomorphism of M with itself), then there is a

decomposition ζ_U of M into invariant sets C such that, for each C, the induced automorphism U_C is ergodic. [In connection with these last two results see Halmos, Duke Math. J. **8**, 386–392 (1941); these Rev. **3**, 50.]

In terms of the concepts defined above, the author's main results may be stated as follows. (1) The isomorphism type of the sequence $\{m_n(C)\}$ of functions on M/ζ constitutes a complete set of isomorphism invariants of the decomposition ζ. (2) The function $\Phi(C)=\tau(U_C)$ from M/ζ_U to the class of all (ergodic) isomorphism types constitutes a complete set of isomorphism invariants of the automorphism U. The author's presentation of both definitions and theorems is very condensed and somewhat unclear; there are no proofs.

P. R. Halmos (Princeton, N. J.).

Zahorski, Zygmunt. Sur l'ensemble des points de non-dérivabilité d'une fonction continue. Bull. Soc. Math. France **74**, 147–178 (1946).

The paper appeared in Russian in Rec. Math. [Mat. Sbornik] N.S. **9**(51), 487–510 (1941); these Rev. **3**, 73.

Radó, Tibor. Two-dimensional concepts of bounded variation and absolute continuity. Duke Math. J. **14**, 587–608 (1947).

Let R be a Jordan region in the $w=u+iv$ plane, $z=t(w)$, $w\epsilon R$, $z=x+iy$, a continuous function in R; let T be the continuous mapping $z=t(w)$, $w\epsilon R$, that transforms R into a set $T(R)$ of the z-plane. For any point z, $N(z, T, R)$ (multiplicity function) denotes the number (possibly ∞) of the points for which $t(w)=z$. Then T is called BV [Banach] if $N(z, T, R)$ is L-integrable in a large enough square of the z-plane. Let $T:z=t(w)$, $w\epsilon R$; $T_*:z=t_*(w)$, $w\epsilon R$, be two continuous mappings and $S(T, T_*)=\max|t(w)-t_*(w)|$, $w\epsilon R$. For every integer $k\geqq 0$ let a set $\mathfrak{R}(k, T, R)$ be defined in the z-plane as follows. A point z belongs to $\mathfrak{R}(k, T, R)$ if and only if there exists an $\epsilon>0$ such that we have $N(z, T, R)\geqq k$ for every continuous mapping $T_*:z=t_*(w)$, $w\epsilon R$, which satisfies the inequality $S(T, T_*)<\epsilon$. In the z-plane the essential multiplicity function $\kappa(z, T, \mathfrak{R})$ is defined as follows: $\kappa(z, T, \mathfrak{R})=k$ if $z\epsilon\mathfrak{R}(k, T, R)-\mathfrak{R}(k+1, T, R)$, $=\infty$ if $z\epsilon\mathfrak{R}(\infty, T, R)$. These concepts were introduced by the author [cf. Math. Ann. **100**, 445–479 (1928)]. If $\kappa(z, T, R)$ is L-integrable, then T is termed eBV [Radó] (essential bounded variation). Radó and Reichelderfer have used these concepts in their study of the transformation of double integrals and the area of surfaces [Trans. Amer. Math. Soc. **49**, 258–307 (1941); **53**, 251–291 (1943); these Rev. **2**, 257; **4**, 213].

Let $\gamma\subset R$ be a simple Jordan region, c the boundary of γ (oriented counterclockwise), C the closed continuous curve of the z-plane into which T transforms c, $O(z, C)$ the topological index [Kronecker] of the curve C with respect to the point z [$O(z, C)=0$ if z belongs to C]. For every z let us put $\Psi(z, T, R)=\sup\sum|O(z, C_i)|$ for any finite group $\{\gamma_i\}$ of simple Jordan regions $\gamma_i\subset R$ without common interior points. The function $\Psi(z, T, R)$ is termed the characteristic function of the continuous mapping T [Cesari, Boll. Un. Mat. Ital. (2) **4**, 109–117, 224–235 (1942); **5**, 5–10 (1943); these Rev. **7**, 513]. If $\Psi(z, T, R)$ is L-integrable, then T is termed BVC [Cesari]. The reviewer used these concepts in his theory [cf. Atti Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. **12**, 1305–1397 (1942); **13**, 1323–1481 (1943); **14**, 891–951 (1944); Ann. Scuola Norm. Super. Pisa (2) **10**, 253–295 (1941); **11**, 1–42 (1942); these Rev. **8**, 142, 257] of the Lebesgue area of surfaces [he has proved also that a surface S has finite Lebesgue area if and only if the three projections of S upon the coordinate planes are BVC].

The author proves in the present paper that $\Psi(z, T, R)=\kappa(z, T, R)$ for every z with countable exceptions and so it is also proved that the concepts eBV and BVC are equivalent. The author then discusses one of the concepts AC used by Radó and Reichelderfer and denoted by ACE in their theory [in the present paper eAC, essential absolute continuity]. The author also discusses the AC concept used by Cesari [ACC] and proves that the concepts eAC and ACC are equivalent. By means of these results it becomes possible to compare the two theories, but this is beyond the scope of the paper under review. *L. Cesari* (Bologna).

Federer, Herbert. The (φ, k) rectifiable subsets of n-space. Trans. Amer. Math. Soc. **62**, 114–192 (1947).

A. S. Besicovitch has studied the geometric properties of plane sets of finite Carathéodory linear measure (density, rectifiability, directionality, restrictedness, projections) and these studies were extended by A. P. Morse and J. F. Randolph. The corresponding problems for two-dimensional measures over three-dimensional space are connected with the theory of surface area. This paper contains a discussion of these properties for a large class of k-dimensional (outer) measures over n-dimensional space, including the measures of Carathéodory, Hausdorff, Gross, Favard (integral geometric measure), and the measure defined by the author.

If φ is a measure over n-space E_n, $A\subset E_n$, $x\epsilon E_n$, $K(x, r)$ is the open sphere with center x and radius r, and $\alpha(k)$ is the volume of the unit sphere of E_k, then

$$\limsup_{r\to 0}\ \varphi(A\cap k(x, r))/\alpha(k)r^k$$

is called the upper (φ, k) density of A at x.

To each $(n-k)$-dimensional flat space F through x and each $\epsilon>0$ corresponds the set $C(x, F, \epsilon)$ of all those points whose distance from F is less than ϵ times the distance from x. The set A is said to be (φ, k)-restricted at x if and only if the upper (φ, k)-density of A at x is positive and there exists an $(n-k)$-dimensional flat space F through x and an $\epsilon>0$ such that the set $(A\cap C(x, F, \epsilon))$ has upper (φ, k) density zero at x.

A subset of E_n is said to be k rectifiable if and only if it is the image of a subset of E_k by a Lipschitz mapping. If $A\subset E_n$ and φ is a measure over E_n, then A is (φ, k) rectifiable if and only if for each $\epsilon>0$ there is a k-rectifiable subset B of A such that $\varphi(A-B)<\epsilon$. The set A is said to be positively (φ, k) unrectifiable if and only if $\varphi(A)>0$ and there is no k rectifiable subset B of A for which $\varphi(B)>0$.

Let p_n^k be the function on E_n to E_k such that $p_n^k(x_1, \cdots, x_n)=(x_1, \cdots, x_k)$ for $(x_1, \cdots, x_n)\epsilon E_n$. To each orthogonal transformation R of E_n corresponds, by superposition, the projection p_n^kR of E_n into E_k, and the Haar measure of the orthogonal group (for which the whole group has measure one) induces a measure over the set of all projections of E_n into E_k, giving meaning to the phrase "almost all projections of E_n into E_k."

Assuming that φ is a measure over E_n, all closed subsets of E_n are φ measurable (in the sense of Carathéodory), $A\subset E_n$, A is a Borel set, and $\varphi(A)<\infty$, some of the main general results can be stated as follows. (1) If the upper (φ, k) density of A is finite at φ almost all points of A, then A is (φ, k) rectifiable if and only if A is (φ, k) restricted at φ almost all of its points. (2) If $\varphi(A)>0$ and the upper (φ, k) density of A is positive at all points of A and is finite at φ almost all points of A, then A is positively (φ, k) unrectifiable if and only if the images of A by almost all

projections of E_n into E_k have k-dimensional Lebesgue measure zero. (3) If the k-dimensional Lebesgue measure of every projection image of each subset of E_n is less than or equal to its φ measure (all the measures mentioned in the first paragraph except the Favard measure satisfy this condition), then A is the union of three disjoint Borel sets A_1, A_2, A_3 such that A_1 is (φ, k) rectifiable, either A_2 is positively (φ, k) unrectifiable or $\varphi(A_2)=0$, and at each point of A_3 the upper (φ, k) density of A is either zero or infinite. (4) If φ is Hausdorff's k-dimensional measure, then $\varphi(A)$ is greater than or equal to the integral geometric Favard measure of A, and a necessary and sufficient condition for equality is that A is (φ, k) rectifiable. These are only a few of the theorems given by the author. For many other characteristic properties of (φ, k) rectifiable and positively (φ, k) unrectifiable sets, for instance the generalization of Besicovitch's results on condensation lines of the first and second kind, we must refer to the paper. The methods employed are only partially connected with those of Besicovitch, Morse and Randolph and with the author's previous work.

The author applies the preceding theory to the problem of nonparametric surface area in E_3. He proves that all the measures mentioned in the first paragraph of this review agree with the Lebesgue area. From this it follows that every nonparametric surface of finite area has an approximate tangent plane at almost all of its points (for instance in the sense of Hausdorff measure).

The paper contains an extensive bibliography.

L. Cesari (Bologna).

***Lorentz, G. G. Über den Gaussschen Integralsatz.** Ber. Math.-Tagung Tübingen 1946, pp. 94–96 (1947).

on order

It is stated that the Gauss theorem applies to any open set of finite volume whose boundary has finite surface measure (for instance, in the sense of Hausdorff) and such that an exterior normal exists (in the classical sense) at all points of the boundary with the possible exception of a set of surface measure zero. [It should be pointed out that the reviewer has proved that all assumptions about the existence of exterior normals are superfluous: Trans. Amer. Math. Soc. **58**, 44–76 (1945); **59**, 441–466 (1946); these Rev. **7**, 199, 422, and the paper reviewed above.] *H. Federer.*

Lorentz, G. G. Beweis des Gaussschen Integralsatzes. Math. Z. **51**, 61–81 (1947).

Proofs of the theorems stated in the paper reviewed above. *H. Federer* (Providence, R. I.).

Theory of Functions of Complex Variables

***Carathéodory, Constantine. A proof of the first principal theorem on conformal representation.** Studies and Essays Presented to R. Courant on his 60th Birthday, January 8, 1948, pp. 75–83. Interscience Publishers, Inc., New York, 1948. \$5.50.

have

The author proves that the unit circle can be mapped one:one and conformally onto the Riemann covering surface over any bounded domain. The proof brings out the topological concepts involved and the method can be used to prove the existence of the surface. The underlying function-theoretic ideas are similar to those used in the author's book [Conformal Representation, Cambridge University Press, 1932]. The reviewer notes the following errors: p. 77, l. 6: read $w=w_1$ for $w\ w_1$; l. 3: read a_1^2, $w_1=0$, A_1 for a^2, $w\ 0$, A, respectively; l. 2: read A_1 for A; p. 81, ll. 12, 5: read notation for rotation. *W. K. Hayman* (Exeter).

Breusch, Robert. On the sum of the relative extrema of $|f(z)|$ on the unit circle. Bull. Amer. Math. Soc. **53**, 982–986 (1947).

Es sei $f(z)=z^n+a_1z^{n-1}+\cdots+a_n$ ein Polynom n-ten Grades, dessen sämtliche Nullstellen im Kreise $|z|\leqq 1$ gelegen sind; $z_\mu=e^{i\theta_\mu}$ $(\mu=1, \cdots, k)$ seien die Punkte, in denen $|f(e^{i\theta})|$ ein relatives Extremum erreicht. Verf. beweist, dass für alle hinreichend grossen n, $\sum_{\mu=1}^{k}|f(z_\mu)|\leqq 2^n$ ist und dass das Gleichheitszeichen nur für $f=(z-e^{i\theta_0})^n$ gelten kann. Es wird für das n keine bestimmte untere Schranke angegeben. Doch zeigen die Beispiele $f=(z+1)(z-\frac{1}{2})$ oder $(z+1)^2(z-\frac{1}{2})$, dass der Satz für kleine n nicht mehr gültig bleibt.

A. Pfluger (Zürich).

Erdös, Paul, and Piranian, George. Over-convergence on the circle of convergence. Duke Math. J. **14**, 647–658 (1947).

Die Arbeit stellt eine verfeinerte Untersuchung in Richtung des Ostrowski'schen Lückensatzes dar. Es wird im Wesentlichen das folgende Theorem bewiesen. Die Potenzreihe $f(z)=\sum_{n=0}^{\infty}a_nz^n$ habe den Konvergenzradius 1 und es bezeichne $\{m_i, n_i\}$ eine Folge von Lücken, d.h. $a_n=0$ für $m_i<n<n_i$, $i=1, 2, \cdots$. Gelten für eine positive konkave Funktion $\varphi(n)$ mit $\varphi(n)=O(\log n)$ die Beziehungen $|a_n|\leqq e^{\varphi(n)}$, $n=1, 2, \cdots$, und $(n_i-m_i)/\varphi(m_i)\rightarrow\infty$, $i\rightarrow\infty$, so ist die Folge $s_{m_i}(z)=\sum_{n=0}^{m_i}a_nz^n$ auf $|z|=1$ lokal gleichmässig konvergent in jedem Punkt, in den sich die gegebene Potenzreihe unmittelbar analytisch fortsetzen lässt. In einem noch allgemeinern Resultat kann von der Bedingung $\varphi(n)=O(\log n)$ abgesehen werden. Besonderer Beachtung wert sind die Spezialfälle $\varphi(n)=c$ und $\varphi(n)=c\log n$, $c=$konstant. Wegen $\varphi(n)=o(n)$ ist die obige Lückenbedingung in derjenigen von Ostrowski $(\liminf_{i\rightarrow\infty} n_i/m_i>1)$ enthalten. Die methodischen Grundgedanken sind dieselben wie in dem M. Riesz'schen Beweis [Ark. Mat. Astr. Fys. **11**, no. 12 (1916); vgl. Landau, Darstellung und Begründung einiger neuerer Ergebnisse der Funktionentheorie, 2. Aufl., Springer, Berlin, 1929] eines Satzes über Potenzreihen, deren Koeffizienten gegen 0 konvergieren; die Verfasser wollen damit einen irrtümlichen, in der Arbeit gegebenen, Literaturnachweis hier berichtigen. *A. Pfluger* (Zürich).

Rakovič, K. Inequalities for absolute values and for coefficients of certain regular function. Acta Fac. Nat. Univ. Carol., Prague. no. **172** (1939), 28–31 (1946). (Czech. English summary)

The author presents a review of inequalities for functions of different types all of which are analytic in $|z|<1$. Besides bounds for the absolute values of the functions, bounds for the coefficients of their power series are considered, their interdependence discussed and ways of improving them indicated. The classes considered include univalent and p-valent functions, functions not assuming certain values, and convex functions. *František Wolf* (Berkeley, Calif.).

Buck, R. Creighton. Interpolation and uniqueness of entire functions. Proc. Nat. Acad. Sci. U. S. A. **33**, 288–292 (1947).

The author considers a sequence T_n of linear functionals defined on the class K of entire functions of exponential

type. In particular, the following questions are raised. For what subclass C of K do the conditions $f\epsilon C$, $T_n(f)=0$ (all n) imply $f(z)\equiv 0$? When is it possible to expand $f(z)$ into a series of the form $f(z)=\sum_{(n)}u_n(z)T_n(f)$? Certain aspects of these problems were discussed in a previous paper [Duke Math. J. 13, 541–559 (1946); these Rev. 8, 371]. Here, the methods used there [due essentially to Pólya and Carlson] are given in a more general form, and are applied to the solution of the expansion problem. Most of the known results on convergence of Abel, Newton and Stirling interpolation series and others are obtained at once as well as the uniqueness theorems of Gelfond. The general method is outlined briefly and a few of the specific results are stated; detailed proofs are to appear later. *A. Pfluger* (Zurich).

Clément, Lucette. Sur une fonction entière qui se présente en calcul symbolique. C. R. Acad. Sci. Paris **225**, 788–790 (1947).

L'auteur étudie les zéros de la fonction entière

$$F(z)=\int_0^h \Gamma(s+2)e^{z(s+1)}ds, \qquad h>0.$$

S. Mandelbrojt (Paris).

Pentikäinen, Teivo. Über stetige Funktionensysteme mit einem algebraischen Additionstheorem. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **38**, 49 pp. (1947).

Let $f_1(u)$, $f_2(u)$ be functions of the real variable u, continuous on an interval $(0, T)$, which are such that $f_1(u+v)$ and $f_2(u+v)$ are algebraic in $f_1(u)$, $f_2(u)$, $f_1(v)$, $f_2(v)$. It is shown that $(0, T)$ can be divided into a finite number of intervals in which the f_i are analytic. In each subinterval, the f_i are obtained from Abelian functions of two variables, or from degenerescences of such functions, by replacing each of the variables by a linear function of u. The relations among the various analytic components of the f are examined. *J. F. Ritt* (New York, N. Y.).

af Hällström, Gunnar. Über Substitutionen, die eine rationale Funktion invariant lassen. Acta Acad. Aboensis **15**, no. 6, 44 pp. (1946).

af Hällström, Gunnar. Zur Reduzibilität der Automorphiefunktionen rationaler Funktionen. Acta Acad. Aboensis **15**, no. 8, 8 pp. (1946).

The author continues investigations by Marty [Ann. Sci. École Norm. Sup. (3) **53**, 83–123 (1936) and papers quoted there] and Shimizu [Tôhoku Math. J. **38**, 219–224 (1933) and papers quoted there] on rational functions $f(z)$ which possess substitutions $f[S(z)]=f(z)$. If $f(z)=P(z)/Q(z)$, where both $P(z)$ and $Q(z)$ are polynomials, the functions S are obviously the solutions of the algebraic equation

$$R(z, S)=\frac{P(S)Q(z)-P(z)Q(S)}{S-z}=0.$$

In order to determine the substitutions $S(z)$ belonging to a given $f(z)$, it is therefore essential to find criteria for the reducibility or irreducibility of $R(z, S)$. Many such criteria are given. If $S(z)$ is a linear substitution, the $f(z)$ reduce to Schwarz's automorphic rational functions (functions of the icosahedron, etc.). Functions $f(z)$ which possess a two-valued $S(z)$ are called by the author "half-automorphic" and similar terms are used in the case of higher algebraic order of $S(z)$. The author shows that, apart from trivial transformations, the only half-automorphic polynomials are the Chebyshev polynomials $\cos(n \cos^{-1} z)$.

In the second paper the author adds some more reducibility criteria for the polynomials $R(z, S)$; these criteria are deduced by means of Riemann's relation connecting the genus and the number of sheets and branch-points of a closed Riemann surface. *Z. Nehari* (Cambridge, Mass.).

***af Hällström, Gunnar. On the study of algebraic functions of automorphism by help of graphs.** C. R. Dixième Congrès Math. Scandinaves 1946, pp. 97–107. Jul. Gjellerups Forlag, Copenhagen, 1947. hmz

Generalizing his work on rational functions [see the preceding review], the author considers meromorphic functions $f(z)$ satisfying functional equations of the type $f(S(z))=f(z)$, where $S(z)$ is an algebraic function. If $S(z)$ is not a linear substitution, i.e., $f(z)$ is not an automorphic function in the normal sense, the totality of substitutions $S(z)$ belonging to a particular $f(z)$ will not form a group, since the functions $S(z)$ are not uniform and the composition of two different substitutions cannot be defined in a unique manner. It may, however, happen that in a suitably cut-up z-plane all the functions $S(z)$ possess uniform branches which do form a group, a necessary and sufficient condition being, of course, that these branches transform the cut-up plane into itself. Since this criterion is not of much practical use, the author develops an alternative method, based on purely topological considerations, which in many cases allows one to decide whether or not a given function $f(z)$ possesses substitutions $S(z)$ which form a group (in the above sense). In these cases it is further possible to determine the degree of the algebraic functions $S(z)$. The main tool used is Speiser's representation of a Riemann surface by means of a graph. The method is therefore only applicable to rational functions or to transcendental meromorphic functions which possess a not too complicated graph. *Z. Nehari* (Cambridge, Mass.).

Hua, Loo-keng. On the extended spaces of several complex variables. Acad. Sinica Science Record **2**, 5–8 (1947).

If the ordinary space of several complex variables is "closed" by addition of "points at infinity" then the question arises as to the totality of complex homeomorphisms then available. The author asserts that, in the case of certain matrix spaces, the algebraically obvious homeomorphisms are analytically the only ones occurring.

S. Bochner (Princeton, N. J.).

Kryloff, N. M. Sur les complexes de Galois. C. R. (Doklady) Acad. Sci. URSS (N.S.) **55**, 683–684 (1947).

Der Verfasser nennt "complexe de Galois" jeden Ausdruck von der Form $\sum_{k=0}^{n-1}\alpha_k i^k$, wo i eine algebraische Zahl vom Grade n ist. Er versucht, ohne dass seine Idee ganz klar würde, die Definition der analytischen Funktionen einer komplexen Variabeln auf Funktionen eines "complexe de Galois" auszudehnen. *W. Nef* (Fribourg).

Kryloff, N. M. Sur les quaternions de W. R. Hamilton et la notion de la monogénéité. C. R. (Doklady) Acad. Sci. URSS (N.S.) **55**, 787–788 (1947).

Bekanntlich haben die analytischen Funktionen einer komplexen Variabeln die Eigenschaft, dass ihre Ableitung von der Richtung unabhängig ist. Der Verfasser schlägt, in übrigens fehlerhafter Weise, vor, Funktionen einer Quaternionenvariabeln zu definieren, welche dieselbe Eigenschaft haben. Er zeigt, dass solche Funktionen eine System von partiellen Differentialgleichungen erfüllen, das den Cauchy-Riemannschen ähnlich ist. Diese Funktionen wur-

den allerdings schon früher von Ringleb [Rend. Circ. Mat. Palermo 57, 311–340, 476–477 (1933)] definiert und von Nisigaki [Tôhoku Math. J. 45, 73–102 (1938)] untersucht. Sie sind alle linear. *W. Nef* (Fribourg).

Theory of Series

Åkerberg, Bengt. On some inequalities. Ark. Mat. Astr. Fys. 34B, no. 13, 3 pp. (1947).

Let $\sum_1^\infty a_n$ be a convergent series of positive terms. The author proves the following two theorems:

$$\text{(1)} \qquad \lim_{n\to\infty} n(a_1a_2\cdots a_n)^{1/n}=0;$$

$$\text{(2)} \qquad \sum_{n=1}^{\infty}(n!a_1a_2\cdots a_n)^{1/n}/(n+1)<\sum_{n=1}^{\infty}a_n,$$

with a corresponding result for finite sums. Since

$$(n!)^{1/n}>(n+1)/e,$$

(2) implies Carleman's well-known inequality

$$\sum(a_1\cdots a_n)^{1/n}<e\sum a_n.$$

[The reviewer remarks that (1) follows by the inequality between arithmetic and geometric means from

$$n^{-1}(a_1+2a_2+\cdots+na_n)\to 0;$$

and that (2) follows in the same way from the identity $\sum_{n=1}^\infty(a_1+2a_2+\cdots+na_n)/[n(n+1)]=\sum_{n=1}^\infty a_n$, as was observed by Knopp, J. London Math. Soc. 3, 205–211 (1928).] *R. P. Boas, Jr.* (Providence, R. I.).

Erdös, Paul, and Piranian, George. A note on transforms of unbounded sequences. Bull. Amer. Math. Soc. 53, 787–790 (1947).

The authors discuss certain aspects of the behavior of unbounded sequences under regular Toeplitz matrix transformations. If A is a regular matrix, then there exist sequences $\{s_n\}$ whose transforms $\{t_n\}$ by A have no finite limit points; if A is row finite, then $\{s_n\}$ may be chosen so that $|t_n|$ becomes unbounded with arbitrary rapidity. In the other direction, it is shown that, given any real function f such that $\lim f(n)=\infty$, there is a regular matrix A such that if $\{s_n\}$ is in the domain of A, and its transform is $\{t_n\}$, then $|t_n|<f(n)$ for infinitely many n. *R. C. Buck.*

Vermes, P. On γ-matrices and their application to the binomial series. Proc. Edinburgh Math. Soc. (2) 8, 1–13 (1947).

The author studies series-to-sequence transformations of the form $G(\sum c_n)\to\sigma$, where the nth element of the sequence σ is determined from the series $\sum c_n$ (and from the elements of the matrix $G=\|g_{nk}\|$ that represents the transformation) according to the rule $\sigma_n=\sum_{k=1}^\infty g_{nk}c_k$. The matrix G is a γ-matrix provided convergence of the series $\sum c_n$ implies convergence of the sequence σ to the same limit. The pth diminutive $G^{(p)}$ of a γ-matrix G is defined to be the matrix G with the first p columns deleted and the remaining columns relabeled. A pth extension G^{+p} of a γ-matrix G is obtained similarly by adjoining p columns at the left (the extension is proper if the extended matrix is also a γ-matrix). The stretched matrix $G^{p\times}$ is obtained from G by repeating p times each column of G. A γ-matrix G is defined to be semi-regular (regular) if convergence of the transform $G\sum c_k$ implies (implies and is implied by) convergence of the transform $G^{(1)}\sum c_k$. Several other concepts and operations on γ-matrices are introduced and their mutual relationship is illustrated by examples. Special reference is made to summability of the binomial series and to the construction of γ-matrices that sum the binomial series for $(1-z)^{-p}$ at an arbitrarily prescribed finite set of points z_k outside of the circle $|z|=1$, and nowhere else outside of this circle. *G. Piranian* (Ann Arbor, Mich.).

Estermann, T. Elementary evaluation of $\zeta(2k)$. J. London Math. Soc. 22, 10–13 (1947).

The author gives an evaluation of $\zeta(2k)$ $(k=1,2,3,\cdots)$ independent of the theory of (real or complex) functions. The result $\zeta(2)=\frac{1}{6}\pi^2$ is derived from $1-\frac{1}{3}+\frac{1}{5}-\cdots=\frac{1}{4}\pi$ without even using double series. Using only absolutely convergent double series, a reduction formula expressing $\zeta(2k+2)$ in terms of $\zeta(2),\zeta(4),\cdots,\zeta(2k)$ is proved. In an addendum it is pointed out that an elementary proof of a similar reduction formula was given by Titchmarsh [Proc. London Math. Soc. (2) 26, 1–11 (1926)]. *N. G. de Bruijn* (Delft).

Proskuryakov, A. P. On the calculation of certain sums in the theory of Hill's equation. Akad. Nauk SSSR. Prikl. Mat. Meh. 11, 563–564 (1947). (Russian)

The author evaluates the sum

$$s=\sum_{n=-\infty}^{\infty}\frac{F(n)}{[(n+k_1)^2-a]\cdots[(n+k_p)^2-a]}=\sum_{n=-\infty}^{\infty}a_n,$$

where $F(n)$ is a polynomial of degree not greater than $2p-2$, by resolving the nth term into partial fractions,

$$a_n=\sum_{1\le l\le p}(A_ln+B_l)/[(n+k_l)^2-a]$$

and using the well-known expression

$$\sum_{n=-\infty}^{\infty}(n^2-a)^{-1}=-\pi a^{-\frac{1}{2}}\cot\pi a^{\frac{1}{2}}.$$

R. Bellman (Princeton, N. J.).

Kullback, S. On the Charlier type B series. Ann. Math. Statistics 18, 574–581 (1947).

The author gives a short proof of the following theorem (not essentially new except for notation). Let $p(r)$ be a function of the nonnegative integral variable r, $\sum p(r)=1$, $\sum|p(r)|<\infty$. Let $\mu_{(0)}=1$, $\mu_{(n)}=\sum_{r=0}^\infty r(r-1)\cdots(r-n+1)p(r)$, $n>0$; $L_n=\sum_{\nu=0}^n(-1)^{n-\nu}\lambda^{n-\nu}\binom{n}{\nu}\mu_{(\nu)}$. Then $p(r)$ can be represented by the absolutely convergent series

$$p(r)=\sum_{n=0}^{\infty}L_nD^n(e^{-\lambda}\lambda^r/r!),\qquad D=\partial/\partial\lambda,$$

if and only if $\sum\mu_{(n)}/n!$ converges. [To the bibliography add H. L. Selberg, Skand. Aktuarietidskr. 25, 228–246 (1942); these Rev. 7, 292. The theorem seems to require a further hypothesis in addition to the tacit one of the convergence of the series defining $\mu_{(n)}$, since the proof depends on the incorrect assumption that a power series which converges uniformly and absolutely on a circle defines a function which is analytic in a larger circle.]

As an application the author establishes the symbolic formula $p(r)=\mu^re^{-\mu}/r!$, where μ^n is to be replaced by $\mu_{(n)}$ after expansion. Some examples are given to illustrate the results of the paper. *R. P. Boas, Jr.* (Providence, R. I.).

Fourier Series and Generalizations, Integral Transforms

Hsiang, Fu-Cheng. The summability $(C, 1-\epsilon)$ of Fourier series. II. Acad. Sinica Science Record **2**, 46–54 (1947).

In part I [Duke Math. J. **13**, 43–50 (1946); these Rev. **7**, 293] it was proved that if

$$(1) \qquad \tfrac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n \cos nx+b_n \sin nx)\sim f(x)$$

and $2\Phi(u)=f(x+u)+f(x-u)-2f(x)$, then the convergence of (2) $\int_0^t\Phi(u)u^{-1-\eta}du$ for some $\eta>0$ is not sufficient for the summability $(C, 1-\epsilon)$ of (1) at the point x if $\epsilon>\eta/(1+\eta)$. It is now established that if $\epsilon=\eta/(1+\eta)$ the condition is sufficient. It is also shown that $\int_0^t\Phi(u)du=O(t/\log(1/t))$ as $t\to0+$ is not sufficient for the summability $(C, 1-\epsilon)$ of (1) at x for any ϵ, $0<\epsilon<1$, but is sufficient for summability $(C, 1)$. *P. Civin* (Eugene, Ore.).

Natanson, I. P. On the summability of the Fourier series of a function of bounded variation. Doklady Akad. Nauk SSSR (N.S.) **57**, 13–15 (1947). (Russian)

Let $K_n(t)$ be a continuous and continuously differentiable function of period 2π, and such that $\lim_{n\to\infty}\int_{-\alpha}^{\beta}K_n(t)dt=1$ for every $0<\alpha\leqq\pi$, $0<\beta\leqq\pi$. Let $f_n(x)=\int_{-\pi}^{\pi}f(t)K_n(t-x)dt$. The author proves the following two results. (1) If $\int_{-\pi}^{\pi}|K_n(t)|dt=O(1)$, then the total variation over $(0, 2\pi)$ of f_n-f approaches 0 as $n\to\infty$. (2) Suppose that, for every $f(x)$ of bounded variation and with regular discontinuities, $\lim_{n\to\infty}f_n(x)=f(x)$, and that $\int_{-\pi}^{\pi}|K_n(t)|dt\to1$. Then the total variation of f_n over $(0, 2\pi)$ tends to f. Both results are well known in the case when $K_n(t)$ is a positive kernel (like Fejér's or Poisson's). [Ad (1) see A. Plessner, J. Reine Angew. Math. **160**, 26–32 (1929), in particular, p. 28; ad (2), Evans, The Logarithmic Potential, Amer. Math. Soc. Colloquium Publ., v. 6, New York, 1927, p. 39.]

A. Zygmund (Chicago, Ill.).

Zygmund, A. On the summability of multiple Fourier series. Amer. J. Math. **69**, 836–850 (1947).

Let $f(x, y)$ be a Lebesgue integrable function of period 2π in both x and y. Let $\sigma_{m,n}(x, y; f)$ be the first (rectangular) Cesàro mean of the Fourier series of f, so that

$$\sigma_{m,n}(x, y; f)=\sum_{\alpha,\beta=-m,-n}^{m,n}\left(1-\frac{|\alpha|}{m+1}\right)\left(1-\frac{|\beta|}{n+1}\right)C_{\alpha\beta}e^{i(\alpha x+\beta y)},$$

where $\{C_{\alpha\beta}\}$ are the Fourier coefficients. The author's main theorem, which generalizes a result of Marcinkiewicz and Zygmund [Fund. Math. **32**, 122–132 (1939), p. 124], is as follows: if $m=m(t)$ and $n=n(t)$ are any two nonnegative nondecreasing integer-valued functions of the parameter t, $0\leqq t<\infty$, tending to infinity with t, and λ is any number greater than or equal to 1, then $\sigma_{\mu,\nu}(x, y; f)\to f(x, y)$ almost everywhere, if μ and ν tend to infinity in such a way that $\lambda^{-1}m(t)\leqq\mu\leqq\lambda m(t)$, $\lambda^{-1}n(t)\leqq\nu\leqq\lambda n(t)$. There are additional theorems in the paper which include extensions of the above result to Abel means and Fourier-Stieltjes series.

K. Chandrasekharan (Princeton, N. J.).

Wintner, Aurel. The Bernoullian Fourier diagrams. Philos. Mag. (7) **38**, 495–504 (1947).

The author discusses the shape of the curves $y=s_\lambda(x)$, $y=c_\lambda(x)$, where

$$s_\lambda(x)=\sum_{k=1}^{\infty}k^{-\lambda}\sin kx, \quad c_\lambda(x)=\sum_{k=1}^{\infty}k^{-\lambda}\cos kx.$$

(i) For $\lambda>0$, the function $s_\lambda(x)$ is positive for $0<x<\pi$. There is a θ_λ, $0<\theta_\lambda<1$, such that $c_\lambda(x)$ is positive inside $(0, \theta_\lambda\pi)$ and negative inside $(\theta_\lambda\pi, \pi)$. (ii) As $\lambda\to+\infty$, we have $\theta_\lambda\to\frac{1}{2}$, $2^\lambda\cos\theta_\lambda\pi\to1$. (iii) Let $0<\lambda<1$. Then both $c_\lambda(x)$ and $s_\lambda(x)$ are decreasing in the interval $0<x<\pi$, and both tend to $+\infty$ as $x\to0$. [See also Fejér, Trans. Amer. Math. Soc. **39**, 18–59 (1936), in particular, pp. 34, 57.]

A. Zygmund (Chicago, Ill.).

Wintner, Aurel. On the shape of the angular case of Cauchy's distribution curves. Ann. Math. Statistics **18**, 589–593 (1947).

The author shows that, if $0<q<1$, $0<\lambda\leqq2$, then the even and periodic function

$$\theta_\lambda(x; q)=1+2\sum_{n=1}^{\infty}q^{n^\lambda}\cos nx$$

is positive and decreasing in the interval $0\leqq x\leqq\pi$. This function is the angular analogue of the density function of the stable laws of P. Lévy. [For $\lambda=2$ the result had been proved by P. Lévy, Bull. Soc. Math. France **67**, 1–41 (1939), in particular, p. 37; these Rev. **1**, 62.] *A. Zygmund.*

Bernšteĭn, S. N. On properties of homogeneous functional classes. Doklady Akad. Nauk SSSR (N.S.) **57**, 111–114 (1947). (Russian)

In a series of papers [see C. R. (Doklady) Acad. Sci. URSS (N.S.) **51**, 331–334, 487–490 (1946); **52**, 563–566 (1946); **54**, 103–108, 475–478 (1946); these Rev. **8**, 20, 323, 373, 509] the author has shown that certain features of the theory of the approximation of periodic functions by trigonometric polynomials persist if we consider the approximation of general functions $f(x)$, $-\infty<x<\infty$, by entire functions $F(x)$ of exponential type p (that is, by functions $F(x)=\sum_0^\infty a_nx^n/n!$, where $\limsup|a_n|^{1/n}\leqq p$). On the other hand, it has recently been shown by the reviewer [see Duke Math. J. **12**, 47–76 (1945); these Rev. **7**, 60] that in the problems of approximation of periodic functions f by trigonometric polynomials, the modulus of continuity (i.e., $\omega(\delta)=\sup|f(x+h)-f(x)|$ for $|h|\leqq\delta$) is sometimes inadequate and has to be replaced by more general expressions (like $\sup|f(x+h)-2f(x)+f(x-h)|$, etc.). The mutual influence of these two facts is discussed in the present paper. Let A_pf denote the best approximation of a function $f(x)$, $-\infty<x<\infty$, by entire functions of exponential type p. Given any $q>0$, let Ω_q be the class of functions f such that $A_pf\leqq M/p^q$ for some $M=M(p)$, and for all $p>0$. Given two sets of numbers $p_1, \cdots, p_k$ and $\alpha_1, \cdots, \alpha_k$ let $S_q=S_q(p_1, \cdots, \alpha_k)$ denote the class of functions $f(x)$, $-\infty<x<\infty$, of finite order (which means that $f(x)=O(|x|^n)$ for $x\to\infty$ and n sufficiently large) and satisfying for all h the inequality $|\sum_{i=1}^kp_i(f(x+\alpha_ih)-f(x))|\leqq M|h|^q$ for some $M>0$. For the sake of simplicity we can always assume that $\sum p_i=1$, $\alpha_1=1<\alpha_2<\cdots<\alpha_k$. The least positive integer $m=m_0$ such that $\sum p_i\alpha_i^m\neq0$ will be called the characteristic of the class. Among others the following results are proved. (1) If $f(x)$ belongs to some $S_q(p_1, \cdots, \alpha_k)$, then $f\epsilon\Omega_q$. (2) If $f(x)$ belongs to some S_q, then f is continuous and has continuous derivatives of all orders $m<q$. (3) If $f\epsilon S_q(p_1, \cdots, \alpha_k)$ and if q exceeds the characteristic m_0 of the class, then f is a polynomial of degree less than m_0. (4) Let m_0 be the characteristic of some $S_q(p_1, \cdots, \alpha_k)$. If $f\epsilon\Omega_q$ and if $q<m_0$, then also $f(x)-G_{n_0}(x)\epsilon S_q$, where G_{n_0} is an arbitrary entire function of exponential type p satisfying an inequality $|f(x)-G_{n_0}(x)|\leqq R/n_0^q$ for some R. (5) If $f(x)$ is bounded over $(-\infty, \infty)$, then the assertion $f\epsilon\Omega_q$ is equiva-

lent to the assertion $f\epsilon S_q(p_1, \cdots, \alpha_k)$, no matter what are the $p_1, \cdots, \alpha_k$, provided the characteristic m_0 exceeds q.

A. Zygmund (Chicago, Ill.).

Geronimus, Ya. L. On best approximation by means of functions not forming a Čebyšev system. Doklady Akad. Nauk SSSR (N.S.) **57**, 7–10 (1947). (Russian)

The author proves the following theorems. (1) Let $f(x)$ and $\varphi_0(x)$ be continuous in $a \leqq x \leqq b$ and let $f(x)-\varphi_0(x)$ attain its maximum absolute value L at (at least) $n+2$ different points of $a \leqq x \leqq b$ with alternating signs; then for every continuous $\varphi(x)$ (not identically equal to $\varphi_0(x)$) such that $\varphi(x)-\varphi_0(x)$ has not more than n zeros, $\max |f(x)-\varphi(x)| > L$; here a zero is counted as single or double according as the function does or does not change sign.

(2) For the function $\varphi_0(x) = a_0 \cos x + b_0 \sin x$ of best approximation in $0 \leqq x \leqq 2\pi$ to a continuous periodic $f(x)$ of period 2π, let $\{x_i\}_1^\mu$ be the points of greatest deviation from $f(x)$; then $\mu \geqq 2$. If $\mu = 2$, it is necessary and sufficient that $x_2 - x_1 = \pi$ and that $f(x)-\varphi_0(x)$ has opposite signs at these points; if $f(x)$ is differentiable at one of these points, $\varphi_0(x)$ is unique. If $\mu = 3$, it is sufficient that $f(x)-\varphi_0(x)$ has alternating signs at the x_i and that $x_3 - x_1 \leqq \pi$; if $x_3 - x_1 < \pi$, $\varphi_0(x)$ is unique. If $\mu \geqq 4$, it is sufficient that $f(x)-\varphi_0(x)$ has alternating signs at the x_i; $\varphi_0(x)$ is always unique.

(3) If $f(x)$ is continuous and has period 2π, and if, in $0 \leqq x \leqq 2\pi$, $|f(x)-\varphi_0(x)| \leqq |f_2(x_1)|$, where

$$\varphi_0(x) = f_1(x_1) \cos (x-x_1) + f_1'(x_1) \sin (x-x_1),$$

$$f_1(x) = \tfrac{1}{2}\{f(x) - f(x+\pi)\}, \quad f_2(x) = f(x) + f(x+\pi), \quad f_2'(x_1) = 0,$$

then $\varphi_0(x)$ is the unique function $a \cos x + b \sin x$ giving the best approximation to $f(x)$ on $0 \leqq x \leqq 2\pi$.

R. P. Boas, Jr. (Providence, R. I.).

Levitan, B. On the approximation to N-almost-periodic functions by finite trigonometric sums. Doklady Akad. Nauk SSSR (N.S.) **56**, 907–910 (1947). (Russian)

The author is responsible for a generalization of the concept of almost periodicity [Comm. Inst. Sci. Math. Méc. Univ. Kharkoff [Zapiski Inst. Mat. Mech.] (4) **15**, 3–35 (1938); Ann. of Math. (2) **40**, 805–815 (1939); V. A. Marčenko, C. R. (Doklady) Acad. Sci. URSS (N.S.) **53**, 7–9 (1946); these Rev. **1**, 53; **8**, 579], the so-called N-almost periodicity. For all positive ϵ and N, there exists a relatively dense set of τ's such that, for $|x| < N$, $|f(x+\tau) - f(x)| < \epsilon$. If, further, τ_1 corresponds to ϵ_1, and τ_2 to ϵ_2, then to $\tau_1 + \tau_2$ should correspond a $\mu(\epsilon_1, \epsilon_2)$ such that $\lim \mu(\epsilon_1, \epsilon_2) = 0$ as $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$. In this paper the author gives a new proof for the approximation theorem by trigonometric polynomials corresponding to this class of almost periodic functions, based on Bochner's theorem [Vorlesungen über Fouriersche Integrale, Leipzig, 1932] on positive definite functions. For Bohr's almost periodic functions this represents a new short proof of the corresponding approximation theorem.

František Wolf (Berkeley, Calif.).

Povzner, A. On the spectrum of bounded functions. Doklady Akad. Nauk SSSR (N.S.) **57**, 755–758 (1947). (Russian)

Let L_1 denote the space of functions integrable in $(-\infty, \infty)$; S, the space of bounded continuous functions in $(-\infty, \infty)$; P, a class of even positive functions $p(x)$ with $p(x)$ continuous and nondecreasing for $x > 0$, $p(\infty) = \infty$, and $p(x+\tau) \leqq M(\tau)p(x)$ for $x > 0$, $\tau > 0$; D_p, the space S with norm $\|\varphi\| = \sup |\varphi(x)/p(x)|$. A linear subspace T of S is called p-invariant if it is closed in D_p and $\varphi(x+\tau)\epsilon T$ if $\varphi(x)\epsilon T$. The author announces the following theorems, with indications of proofs.

(1) If T is a p-invariant proper subspace of S, and if f is an element of S not in T, there exists $h\epsilon L_1$ such that $\int_{-\infty}^{\infty} h(t) f(-t)dt \neq 0$ and $\int_{-\infty}^{\infty} h(t)\varphi(-t)dt = 0$ if $\varphi\epsilon T$.

If $\varphi\epsilon S$, $h\epsilon L_1$, write $h \circ \varphi = \int_{-\infty}^{\infty} \varphi(x-t)h(t)dt$, and denote by $H(T)$ the set of elements of L_1 such that $h\circ\varphi = 0$ for every $\varphi\epsilon T \subset S$. If o is considered as a multiplication in L_1, $H(T)$ is a closed ideal in the ring L_1. (2) If T is a p-invariant space, $\varphi\epsilon S$ and $\varphi \circ h = 0$ for every $h\epsilon H(T)$, then $\varphi\epsilon T$.

The closed linear manifold determined in the D_p metric by $\varphi(x+\tau)$ is called $T_{\varphi, p}$. The set of numbers λ for which $\int_{-\infty}^{\infty} e^{-i\lambda t} h(t)dt = 0$ for all $h\epsilon H(T_{\varphi, p})$ is called the p-spectrum of φ and denoted by $\mathfrak{N}_{\varphi, p}$. (3) If $\lambda\epsilon\mathfrak{N}_{\varphi, p}$ then $e^{i\lambda x}\epsilon T_{\varphi, p}$. (4) The p-spectrum of every $\varphi\epsilon S$ is nonempty. The proof depends on Wiener's Tauberian theorem. (5) The intersection $\mathfrak{N}_\varphi$ of all $\mathfrak{N}_{\varphi, p}$, $p\epsilon P$, is not empty. There is a $q\epsilon P$ such that $\mathfrak{N}_\varphi = \mathfrak{N}_{\varphi, q}$. The set $\mathfrak{N}_\varphi$ is closed; it is called the H-B spectrum of φ. Hence if λ belongs to the H-B spectrum, $e^{i\lambda t}$ belongs to the span of $\{\varphi(t+\tau)\}$ in any D_p.

Let $\mathfrak{N}$ be a closed set of real numbers. The subset of L_1 of functions h with $\int_{-\infty}^{\infty} e^{-i\lambda t} h(t)dt = 0$, $\lambda\epsilon\mathfrak{N}$, is denoted by $H_{\mathfrak{N}}$. The set of λ for which $\int_{-\infty}^{\infty} e^{-i\lambda t}h(t)dt = 0$ for all $h\epsilon J \subset L_1$ is denoted by $\mathfrak{N}(J)$. The set of $h\epsilon L_1$ for which $h\circ\varphi = 0$ is denoted by H_φ. Then $\mathfrak{N}_\varphi = \mathfrak{N}(H_\varphi)$. A-problem: Can a given $\varphi\epsilon S$ be approximated in D_p by a sequence of linear combinations of $e^{i\lambda t}$, $\lambda\epsilon\mathfrak{N}_\varphi$? (6) For the A-problem to have an affirmative solution it is necessary and sufficient that $H_\varphi = H_{\mathfrak{N}_\varphi}$. T-problem: Does a given closed ideal $J \subset L_1$ coincide with $H_{\mathfrak{N}(J)}$? (7) The T-problem has an affirmative solution for every closed ideal J if and only if the A-problem has an affirmative solution for every $\varphi\epsilon S$. From (6) and a theorem of Ditkin [Uchenye Zapiski Moskov. Gos. Univ. Matematika **30**, 83–130 (1939); these Rev. **1**, 336] follows (8): If the boundary of $\mathfrak{N}_\varphi$ is a reducible set, the A-problem has an affirmative solution for φ.

The Hahn-Bochner spectrum of φ is the set of points λ for which there is no neighborhood of λ in which the generalized Fourier transform of φ, of the second order, is linear. (9) The H-B spectrum of φ coincides with the Hahn-Bochner spectrum. The author reports that V. Marčenko has shown that, if $\varphi(x)$ is uniformly continuous, the spectrum in the sense of Beurling [Acta Math. **77**, 127–136 (1945); these Rev. **7**, 61] coincides with the H-B spectrum.

(10) If $\lambda\epsilon\mathfrak{N}_\varphi$ and Δ_λ is a neighborhood of λ, it is impossible to approximate to φ in any D_p by linear combinations of $e^{i\lambda x}$ with λ not belonging to Δ_λ. (11) If Δ is an open subset of $\mathfrak{N}_\varphi$ and the boundary of Δ is reducible, then φ can be approximated in any D_p by linear combinations of $e^{i\lambda x}$ with $\lambda\epsilon\Delta$. (12) For φ to belong to $T_{\psi, p}$ it is necessary and sufficient that $H_\psi \subset H_\varphi$.

R. P. Boas, Jr. (Providence, R. I.).

Povzner, A. On the spectrum of bounded functions and the Laplace transform. Doklady Akad. Nauk SSSR (N.S.) **57**, 871–874 (1947). (Russian)

The author considers continuous functions bounded on the real axis, but states that his results generalize easily to more general functions. He writes, for $\Im(z) < 0$ or $\Im(z) > 0$, respectively,

$$L_F(z) = \int_0^\infty e^{izt} F(t)dt, \quad L_F(z) = \int_0^{-\infty} e^{izt} F(t)dt.$$

He says that $F(x)$ has no spectrum in a certain interval if its second-order generalized Fourier transform $\Phi_F(\lambda)$ (in the

sense of Bochner) is linear there [cf. the preceding review]. He says that $F(x)$ has an analytic spectrum in $\Delta=(\lambda_1, \lambda_2)$ if, for every interior interval $\Delta_\delta=(\lambda_1+\delta, \lambda_2-\delta)$, $F(x)$ can be represented as $F_{1,\delta}+F_{2,\delta}$, where $F_{1,\delta}$ has no spectrum in Δ_δ and $F_{2,\delta}(x)=\int_{\lambda_1+\delta}^{\lambda_2-\delta}\psi(\lambda)e^{ix\lambda}d\lambda$, with $\psi(\lambda)$ analytic in (λ_1, λ_2).

Theorem 1. The following representation is valid:

$$L_F(z)=(2\pi i)^{-1}\int_{-\infty}^{\infty}\Phi_F(\lambda)(z-\lambda)^{-2}d\lambda.$$

This is proved by approximating to $F(x)$ by entire functions of exponential type. Theorem 2. A necessary and sufficient condition for $F(x)$ to have an analytic spectrum in (λ_1, λ_2) is that $L_F(z)$ can be continued analytically across the interval (λ_1, λ_2) of the real axis in the z-plane. For $f(x)$ to have no spectrum in (λ_1, λ_2) it is necessary and sufficient that $L_F(z)$ is regular in the region composed of the upper and lower half planes and the same interval.

R. P. Boas, Jr. (Providence, R. I.).

✱**Doetsch, G. Tabellen zur Laplace-Transformation und Anleitung zum Gebrauch.** Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen, Band LIV. Springer-Verlag, Berlin and Göttingen, 1947. ix+185 pp.

The first part [pp. 1–71] collects, without proofs, the results about Laplace transforms necessary for their application to the solution of differential equations and discusses some typical examples. An example of a heat-conduction problem in which formal computation leads to an incorrect result serves as a warning to those who believe that such things cannot happen in practice. The relationship of the Laplace transform method to operational methods is carefully discussed.

The second part gives the most extensive table yet published of pairs of transforms $f(s)=\int_0^\infty e^{-st}F(t)dt$, arranged according to the character of $f(s)$; in each case the abscissa of convergence is given. A 22-page index of functions, arranged alphabetically by the names and symbols of the functions, enables the table to be used also in the opposite direction. The wide scope of the table can be seen from the following summary. Operations (56 formulas); Rational functions (65); Irrational functions (105); Logarithmic and inverse trigonometric functions (90); Exponential functions (102); Trigonometric and hyperbolic functions (74); Gamma and related functions (37); Exponential and sine integrals, etc. (42); Confluent hypergeometric functions (45); Bessel functions (108); Spherical harmonics (14); Elliptic integrals (28); Theta functions (15); Special functions (12); Hypergeometric and other series (42). *R. P. Boas, Jr.*

Pollard, Harry. The integral transforms with iterated Laplace kernels. Duke Math. J. 14, 659–674 (1947).

Die Funktionen $G_n(x, y)$ seien durch die folgende Gleichung definiert:

$$G_n(x, y)=\int_0^\infty e^{-xt}G_{n-1}(t, y)dy \qquad n=1, 2, \cdots; G_0(x, y)=e^{-xy}.$$

Diese Funktionen zerfallen in zwei Klassen, je nachdem, ob n gerade oder ungerade ist. Der Verf. untersucht das Inversionsproblem für die Transformierte

$$f(x)=\int_{0+}^\infty K_n(x, y)d\alpha(y),$$

wobei $\alpha(y)$ von beschränkter Variation in jedem endlichen Intervall der positiven Achse sein soll und $K_n(x, y)=G_{2n}(x, y)$. Das entsprechende Inversionsproblem für ungerade n wurde vom Verf. bereits früher gelöst [Duke Math. J. 14, 129–142 (1947); diese Rev. 8, 578]. Ueber $\alpha(y)$ stellt der Verfasser gegenüber frühern Untersuchungen keine weitern Bedingungen als die Konvergenz des obigen Integrales.

W. Saxer (Zürich).

Tagamlickiĭ, Ya. A. On absolutely convergent Dirichlet series. Doklady Akad. Nauk SSSR (N.S.) 57, 875–878 (1947). (Russian)

Tagamlickiĭ, Ya. A. On the absolutely convergent Laplace integral. Doklady Akad. Nauk SSSR (N.S.) 58, 197–200 (1947). (Russian)

Necessary and sufficient conditions for $f(x)$ to be an absolutely convergent Laplace integral, or an absolutely convergent Dirichlet series with exponents λ_n, for $x>a$, are that $|f^{(k)}(x)|\leqq(-1)^k g_k(x)$ for $k=0, 1, 2, \cdots$ and $x>a$, where $g(x)$ is, respectively, of the form $\int_0^\infty e^{-xt}G(t)dt$, $G(t)\geqq 0$, or $\sum G_n e^{-\lambda_n x}$, $G_n\geqq 0$. These results were also obtained, respectively, by Loève and Boas in notes inspired by earlier work of the author [see C. R. Acad. Sci. Paris 225, 31–33 (1947); 224, 1683–1685 (1947); 223, 940–942 (1946); these Rev. 9, 82; 8, 569, 259]. *R. P. Boas, Jr.*

Lebedev, N. N. On the representation of an arbitrary function by an integral involving Macdonald functions of complex order. Doklady Akad. Nauk SSSR (N.S.) 58, 1007–1010 (1947). (Russian)

Let $I_\mu(x)$, $K_\mu(x)$ denote the Bessel functions of an imaginary argument. The author establishes the formula

$$f(x)=(\pi i)^{-1}\int_{\sigma-i\infty}^{\sigma+i\infty}\mu K_\mu(x)d\mu\int_0^\infty \xi^{-1}f(\xi)I_\mu(\xi)d\xi,$$

where $\mu=\sigma+i\tau$, $x>0$, $\sigma>1$, subject to the (sufficient) conditions that $f(x)$ is a normalized function of bounded variation in every interval (a, A), $0<a<A$, and $x^{-1}f(x)I_\sigma(x)\epsilon L(0, \infty)$.

R. P. Boas, Jr. (Providence, R. I.).

Dufresnoy, Jacques. Sur le produit de composition de deux fonctions. C. R. Acad. Sci. Paris 225, 857–859 (1947).

Let $f(x)$ and $g(x)$ be continuous in $[0, x_0]$ and let $\int_0^x f(t)g(x-t)dt\equiv 0$ in this interval. Then $f(x)\equiv 0$ in $(0, \xi_0)$ and $g(x)\equiv 0$ in $(0, \xi_0')$, where $\xi_0+\xi_0'\geqq x_0$. [This result, for integrable f and g and "zero almost everywhere" in place of "zero" in hypothesis and conclusion, is proved in Titchmarsh's Introduction to the Theory of Fourier Integrals, Oxford, 1937, p. 327; cf. also M. M. Crum, Quart. J. Math., Oxford Ser. 12, 108–111 (1941); these Rev. 3, 39.] The author's proof proceeds by applying to the Laplace transforms of $f(x)$ and $g(x)$ the following theorem on functions analytic in a half plane. Let $F(z)$ be analytic in $x>0$, continuous and bounded on $x=0$, real for real x and satisfying lim inf $r^{-2}\log M(r)\leqq 0$; let $G(z)$ have the same properties; if $F(z)G(z)$ is bounded, so is one of $F(z)$ and $G(z)$.

R. P. Boas, Jr. (Providence, R. I.).

Polynomials, Polynomial Approximations

de Sz. Nagy, Gyula. Generalization of certain theorems of G. Szegö on the location of zeros of polynomials. Bull. Amer. Math. Soc. 53, 1164–1169 (1947).

This paper deals with certain cases of the problem: given that all the zeros of $f(z)=(z-a_1)(z-a_2)\cdots(z-a_n)$ lie in a

circular region C_1 and that all those of $g(z)=(z-b_1)\cdots(z-b_n)$ lie in a circular region C_2 having no points in common with C_1, to find the location of the zeros of $h(z)=f(z)-Ag(z)$. By elementary methods the author proves principally the following extensions of some theorems of G. Szegő [Math. Z. **13**, 28–55 (1922)]. (I) For C_1: $|z-\alpha|\geqq r_1$ and C_2: $|z-\alpha|\leqq r_2<r_1/|B|$ with $B^n=A$, then $h(z)\neq 0$ in the circle $|z-\alpha|\leqq(r_1-|B|r_2)/(1+|B|)$. (II) For $|A|=1$, C_1: $|z-\alpha|\leqq r_1$ and C_2: $|z-\beta|\leqq r_2$, then $h(z)\neq 0$ in the hyperbola with foci at α and β and transverse axis r_1+r_2. (III) For $|A|\leqq 1$, C_1: $|z-\alpha|\geqq r_1$ and C_2: $|z-\beta|\leqq r_2<r_1$, then $h(z)\neq 0$ in the ellipse with foci at α and β and with major axis r_1-r_2.

Reviewer's note. Reference concerning theorem I should have been made to J. L. Walsh [Trans. Amer. Math. Soc. **24**, 163–180 (1922), p. 171], for a treatment of the case C_1: $|z-\alpha|\leqq r_1$ and C_2: $|z-\beta|\leqq r_2$ and for the suggestion of analogous results in the case of arbitrary circular regions C_1 and C_2. In fact, by Walsh's method and lemma 2 of M. Marden [Bull. Amer. Math. Soc. **42**, 400–406 (1936), p. 402] the more general result may be deduced at once that, if C_1: $|z-\alpha|\geqq r_1$ and C_2: $|z-\beta|\leqq r_2<r_1/|B|$, then $h(z)\neq 0$ in all the circles with centers $(\alpha-B\beta)/(1-B)$ and radii $(r_1-|B|r_2)/|1-B|$ with $B^n=A$ but $B\neq 1$. Further reference should have been made to J. L. Walsh [Amer. Math. Monthly **29**, 112–114 (1922)] for the first statement and proof of theorem II and for the first statement of theorem III. *M. Marden* (Milwaukee, Wis.).

Bilharz, Herbert. Über die Frequenzgleichung bei Stabilitätsuntersuchungen nach der Methode der kleinen Schwingungen. Jahrbuch **1940** der Deutschen Luftfahrtforschung, I565–I574 (1940).

The Routh-Hurwitz criteria that all the zeros of a real polynomial $f(z)=a_0z^n+a_1z^{n-1}+\cdots+a_n$, $a_0>0$, have negative real parts are stated and given the following geometrical interpretations. Let b_j be defined as $b_j=a_{j+1}$ for j even and as $b_j=(a_1a_{j+1}-a_0a_{j+2})/a_1$ for j odd, with $a_k=0$ for $k>n$. When $n=4$, the Routh-Hurwitz criteria are equivalent to the condition $0<\beta<\alpha<\pi/2$ upon the angles α and β made with the x-axis by the vectors $A=(a_1, a_3)$ and $B=(b_1, b_3)$, respectively. When $n=5$, they are equivalent to the condition that the vector $A=(a_1a_2-a_0a_3, a_1a_4-a_0a_5, a_3a_4-a_2a_5)$ has only positive components and lies inside the cone $x_1x_3-x_2^2>0$. When $n=6$, they are equivalent to the condition that the vectors $A=(a_1, a_3, a_5)$ and $B=(b_1, b_3, b_5)$ have only positive components and that their vector product $C=A\times B=(c_1, c_2, c_3)$ has the components $c_1<0$, $c_2>0$ and $c_3<0$ and lies inside the cone $x_1x_3-x_2^2>0$. In the general case, in which the coefficients a_j depend on r parameters, the parameters are considered as the coordinates of a point p in r-space and in that space the two manifolds P: $a_k(p)>0$ $(k=0, 1, \cdots, n)$ and M: $D(p)>0$ are introduced, $D(p)$ being the Routh discriminant of $f(z)$. It is then proved that, if a stable state exists for $p=p_0$, it will continue to exist as p is varied continuously from p_0 provided p remains in the product of P and M. *M. Marden* (Milwaukee, Wis.).

Greville, T. N. E. Remark on the note "A generalization of Waring's formula." Ann. Math. Statistics **18**, 605–606 (1947).

The author reports that the formula established by him [same Ann. **15**, 218–219 (1944); these Rev. **6**, 63] had been given by Hermite [J. Reine Angew. Math. **84**, 70–79 (1877)].

Harmonic Functions, Potential Theory

Kasner, Edward. Note on conjugate harmonic functions. Amer. Math. Monthly **54**, 405–406 (1947).

It is pointed out that the problem of determining a harmonic function $\psi(x, y)$ conjugate to a given harmonic function $\varphi(x, y)$ can be solved without resorting to integration. It can be shown that we have $\varphi(x, y)=\mu(x-iy)+\lambda(x+iy)$, where μ and λ are analytic functions whose corresponding coefficients are conjugate imaginaries; consequently, we have $\psi(x, y)=i(\mu-\lambda)+c$, where c is an arbitrary real constant. An advantage of the present method is that it enables one to give simple proofs of the results that the conjugate of a rational, algebraic or entire harmonic function is rational, algebraic or entire, respectively. *E. F. Beckenbach.*

Makai, E. A property of mean of harmonic functions. Rend. Circ. Mat. Palermo **63**, 33–40 (1942).

This paper is devoted to a new treatment of a known mean value property of harmonic functions. The author cites the following theorem due to Ásgeirsson. In an n-dimensional space S_n let u be harmonic in a domain containing I_1 and I_2, the interiors of two $(n-1)$-dimensional confocal ellipsoids. Then the integral means of u over I_1 and I_2 are equal [Math. Ann. **113**, 321–346 (1937)]. The discussion is based on ellipsoidal harmonics and is given in detail for ellipsoids in S_3 which are not surfaces of revolution. The author includes also a proof for ellipses in S_2, based on conformal mapping. *F. W. Perkins.*

Ahlfors, Lars V. Das Dirichletsche Prinzip. Math. Ann. **120**, 36–42 (1947).

As the author points out, in the present paper he presents a previously published [Soc. Sci. Fenn. Comment. Phys.-Math. **11**, no. 15 (1943); these Rev. **7**, 203] development of the Dirichlet principle, this time avoiding a symbolism which might be unfamiliar to some readers.

E. F. Beckenbach (Los Angeles, Calif.).

Miranda, C. Sul principio di Dirichlet per le funzioni armoniche. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **3**, 55–59 (1947).

L'auteur traite le problème de Dirichlet classique en utilisant une idée de Caccioppoli [Ann. Scuola Norm. Super. Pisa (2) **7**, 177–187 (1938)]. Il suffit de voir, grâce à la théorie de Hahn-Banach sur les fonctionnelles linéaires, que si pour le domaine borné D de frontière S, l'intégrale de Radon $\int_S f d\mu=0$ pour toute f finie continue rendant le problème résoluble, l'intégrale est nulle pour toute fonction finie continue, puis même que la nullité du potentiel de μ (chargeant S) à l'extérieur de D entraine la nullité de μ. C'est ce qui est établi avec des hypothèses de régularité sur S (comme la courbure, etc.). Si cela peut être utile didactiquement ou pour une transposition à des équations de type elliptique, il convient de rappeler que la condition requise sur le potentiel de μ est satisfaite si (et seulement si, pour tous les μ) l'extérieur de D n'est effilé nulle part sur S. Même en écartant toute frontière "intérieure," on voit donc que l'idée utilisée ne peut conduire à la condition générale de résolubilité qui est l'absence de points-frontière où le complémentaire du domaine D est effilé. *M. Brelot.*

Evans, G. C. Multiply valued harmonic functions. Green's theorem. Proc. Nat. Acad. Sci. U. S. A. **33**, 270–275 (1947).

On considère dans l'espace analogue à une surface de Riemann à plusieurs feuillets dans le plan, mais avec trois

dimensions, un domaine borné T dont la frontière est formée de r courbes fermées disjointes s_i non entrecroisées, de capacité nulle et d'une partie "extérieure" T^* formée, sur m feuillets, de m surfaces régulières. Alors si $u(M)$, $v(M)$ sont harmoniques, bornées dans T et au voisinage de tout point de T^*, et si n désigne la normale extérieure en P:

$$\int_{T^*} u(dv/dn)dP = \int_T (\operatorname{grad} u \cdot \operatorname{grad} v)dM,$$

où les s_i ne jouent pas de rôle dans les intégrales.

M. Brelot (Grenoble).

Differential Equations

Alardin, Félix. Sur un théorème d'existence et d'unicité de l'intégrale d'une équation différentielle linéaire. Ann. Soc. Sci. Bruxelles. Sér. I. **61**, 205–214 (1947).

Let $y=y(x)\not\equiv 0$ be a solution of a nonsingular linear homogeneous differential equation of order n. Assuming that the coefficient functions are analytic with uniformly bounded derivatives, the author calculates a lower bound for the length of the interval on which there is at least one zero for each of the functions $y, y', y'', \cdots, y^{(n-1)}$.

P. Hartman (Baltimore, Md.).

Rosenblatt, Alfred. On the unicity of solutions of a system of two ordinary differential equations of the first order satisfying given initial conditions in the real domain. Revista Ci., Lima **49**, 183–200 (1947).

Let $|f(x, y_1, z_1)-f(x, y_2, z_2)| \leqq A|z_2-z_1|x^{-m/p}$ and $|g(x, y_1, z_1)-g(x, y_2, z_2)| \leqq B|y_2-y_1|x^{-n/p}$ hold for all points (x, y_1, z_1), (x, y_2, z_2) in the half-space $x>0$. The author shows that if the nonnegative constants A, B, n, m, p satisfy certain inequalities, then the system of differential equations $dy/dx=f$, $dz/dx=g$ has at most one solution satisfying $y(0)=z(0)=0$.

P. Hartman (Baltimore, Md.).

Abelé, Jean. Construction d'oscillateurs non linéaires sinusoïdaux par la méthode de l'axe mobile. C. R. Acad. Sci. Paris **225**, 1270–1271 (1947).

Fichera, G. Sull'esistenza delle funzioni potenziali nei problemi della fisica matematica. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **2**, 527–532 (1947).

The author states that in a work in progress he has proved the following theorem. Let S be a system of n linear homogeneous partial differential equations with constant coefficients in two independent variables. Then there exist n linear differential operators D_i with constant coefficients, such that, if $u_1, \cdots, u_n$ are a solution of S, there is a function F such that $u_i=D_iF$. The present note is devoted to proving that the corresponding theorem with three independent variables is false. The counterexample exhibited is the system S of equations of equilibrium of an elastic medium.

J. W. Green (Los Angeles, Calif.).

Picone, Mauro. Nouvelles méthodes de recherche pour la détermination des intégrales des équations linéaires aux dérivées partielles. Ann. Soc. Polon. Math. **19** (1946), 36–61 (1947).

Par l'application de la transformation

$$T[f(x,t)] = \int_0^T e^{\lambda(T-t)} f(x,t)dt,$$

que l'auteur appelle transformation de Laplace à intervalle d'intégration fini, aux deuxième membre d'une équation aux dérivées partielles de la forme

$$(1)\qquad a_{20}u_{xx}+2a_{11}u_{xt}+a_{02}u_{tt}+a_{10}u_x+a_{01}u_t+a_{00}u=f(x,t),$$

où les coefficients a_{ij} sont des fonctions de la seule variable x, on passe à une equation différentielle en x de la forme

$$(2)\qquad a_{20}u^*_{xx}+A_1(x,\lambda)u^*_x+A_0(x,\lambda)u^*=V(x,\lambda),$$

dans laquelle $u^*(x, \lambda)$ représente la fonction transformée de $u(x, t)$ et le second membre V comprend des termes dépendants des valeurs de u et de sa dérivée par rapport à t pour $t=0$ et pour $t=T$, dont seulement les premières se supposent connus. Dans le cas particulier dans lequel ces valeurs initiales sont égaux à 0, on démontre, sous des conditions peu restrictives, que la solution de (2) satisfait la condition $|u^*(x, \lambda)| \leqq K\lambda/\mu$, où K et μ sont des constantes. Cette observation permet d'énoncer un certain nombre de théorèmes d'unicité à l'égard de la solution de l'équation (1), sous différentes hypothèses relatives aux coefficients a_{ij} et aux conditions aux limites. Le cas particulier de l'équation du premier ordre est considéré. Le problème de l'existence et du calcul de la solution se trouve beaucoup moins avancé.

B. Levi (Rosario).

Nef, Walter. Die Funktionentheorie der partiellen Differentialgleichungen zweiter Ordnung. (Hyperkomplexe Funktionentheorie). Bull. Soc. Fribourgeoise Sci. Nat. **37** (1942/44), 348–375 (1946).

In a clear presentation the author gives a brief survey of the work of R. Fueter (and that of others based on Fueter's work), in partial differential equations of the second order in n variables. This method has a basic analogy to Heaviside's operational methods in ordinary differential equations with constant coefficients. The technique, however, is quite different. In order to "factor" a homogeneous partial differential equation of the second order in n variables the algebra of Clifford for hypercomplex numbers is introduced. For the hypercomplex function solutions of these equations some results analogous to those in analytic function theory are obtained.

A. Gelbart.

Lyubov, B. Ya. Solution of a nonstationary distribution problem of heat conduction for a region with uniform transmission at the boundary. Doklady Akad. Nauk SSSR (N.S.) **57**, 551–554 (1947). (Russian)

The physical problem reduces to the solution of $T_{xx}=a^{-1}T_t$, subject to $T(x, t)=\phi(t)$ for $x=0$, $t\geqq 0$, $T(x, t)=\psi(t)$ for $x=ct$, $t\geqq 0$, $\phi(0)=\psi(0)=0$. A solution of the form

$$T(x,t)=(\pi a)^{-\frac{1}{2}}\int_0^t \{a(y)e^{-(z-\beta y)^2/(t-y)} + b(y)e^{-z^2/(t-y)}\}(t-y)^{-\frac{1}{2}}dy,$$

$z=\frac{1}{2}xa^{-\frac{1}{2}}$, $\beta=\frac{1}{2}ca^{-\frac{1}{2}}$, is tried. Using the Laplace transform on the equations resulting from the substitution of the boundary conditions, $a(y)$ and $b(y)$ are obtained.

R. Bellman (Princeton, N. J.).

Tranter, C. J. Note on a problem in heat conduction. Philos. Mag. (7) **38**, 530–531 (1947).

A boundary value problem involved in a paper by R. L. Brown [Philos. Mag. (7) **37**, 318–322 (1946); these Rev. **8**, 585] is solved here by means of Laplace transforms.

R. V. Churchill (Ann Arbor, Mich.).

Gårding, Lars. The solution of Cauchy's problem for two totally hyperbolic linear differential equations by means of Riesz integrals. Ann. of Math. (2) 48, 785–826 (1947).

The solution of the problem of Cauchy for two new types of linear totally hyperbolic partial differential equation is obtained by means of a generalisation of the Riesz-Riemann-Liouville integral of fractional order [M. Riesz, C. R. Congrès Internat. Math., Oslo, 1936, v. 2, pp. 44–45]. The ordinary *RRL* integral is defined to be

$$I^p f(x)=\{H_n(p)\}^{-1}\int_{E(x)} f(y)r(x-y)^{p-\frac{1}{2}n}dy,$$

where dy is the element of volume in an n-dimensional Lorentz space, $r(x)$ is the square of the "interval" from 0 to the point x, so that $r(x)=x_1^2-x_2^2-x_3^2-\cdots-x_n^2$; the domain of integration $E(x)$ is bounded by the "retrograde light cone" $r(x-y)=0$, $x_1>y_1$, and a certain hypersurface G. The complex constant p corresponds to $\frac{1}{2}\alpha$ in Riesz's work, and $H_n(p)$ depends only on p. This integral has the properties $I^pI^qf=I^{p+q}f$, $\Delta I^{p+1}f=I^p$, where Δ is the second differential parameter in the Lorentz space.

In Gårding's first generalisation of the *RRL* integral, the Lorentz space is replaced by a space S whose points are real symmetric matrices x of order n; the coordinates of x are the $\frac{1}{2}n(n+1)$ real numbers x_{ij} where $i\geqq j$. If

$$\sum_{i,k} x_{ik}u^iu^k=\sum_{h=1}^{m}(\sum_i a_{hi}u^i)^2,$$

where the real linear forms $\sum_i a_{hi}u^i$ are linearly independent, the point x is said to be positive semi-definite of rank m; in particular, when $m=n$, x is positive definite, and we write $x>0$. The region $x>0$ is an open convex cone whose boundary points are positive semi-definite.

Corresponding to the retrograde light cone with vertex x in Lorentz space, we have now the set of points y in which $x-y>0$. If G is a suitable hypersurface, it cuts off from this set a domain $E(x)$. The generalised *RRL* integral is then

$$I^pf(x)=\{\Gamma_n(p)\}^{-1}\int_{E(x)} f(y)\{\det(x-y)\}^{p-\frac{1}{2}(n+1)}dy,$$

where dy is the volume element $\prod_{i\geqq j}dy_{ij}$ in S, and $\Gamma_n(p)$ is a function of p, which does not depend on the choice of f. This integral has the properties $I^pI^qf=I^{p+q}f$, $DI^{p+1}f=I^pf$, where D is Cayley's differential operator

$$2^{-n}\begin{vmatrix} 2\partial/\partial x_{11} & \partial/\partial x_{21} & \partial/\partial x_{31} & \cdots \\ \partial/\partial x_{21} & 2\partial/\partial x_{22} & \partial/\partial x_{32} & \cdots \\ \partial/\partial x_{31} & \partial/\partial x_{32} & 2\partial/\partial x_{33} & \cdots \\ \cdots & & \cdots & \end{vmatrix}.$$

The integral $I^pf(x)$ is an analytic function of p, regular in $\Re p>\frac{1}{2}(n-1)$, which can be continued analytically if f and the surface G are sufficiently well behaved; and then $I^0f(x)=f(x)$, $I^{-k}f(x)=D^kf(x)$. By the aid of this integral, the problem of Cauchy is solved for $D^mu=h(x)$, where $u(x)$ and its derivatives of order less than mn are given on G. In the simplest case, $n=2$, $m=1$, when

$$D=\frac{\partial^2}{\partial x_{11}\partial x_{22}}-\frac{\partial^2}{4\partial x_{21}^2},$$

which is equivalent to the cylindrical wave operator.

In his second generalisation, Gårding takes a space H whose points are Hermitian matrices $x=x'+ix''$, where x' is real symmetric, x'' real skew. The coordinates are now the numbers x_{ii}', x_{ij}', x_{ij}'' with $i>j$. In much the same way, an operator I^p is defined, where I is now, so to speak, the inverse of the Hermitian operator

$$2^{-n}\begin{vmatrix} 2\partial/\partial x_{11}' & \partial/\partial x_{21}'+i\partial/\partial x_{21}'' & \cdots \\ \partial/\partial x_{21}'-i\partial/\partial x_{21}'' & 2\partial/\partial x_{22}' & \cdots \\ \cdots & \cdots & \end{vmatrix}.$$

As the analysis is rather difficult, it is impossible to go into more detail in a short space. *E. T. Copson* (Dundee).

Karimov, Dž. H. On periodic solutions of nonlinear equations of the fourth order. Doklady Akad. Nauk SSSR (N.S.) 57, 651–653 (1947). (Russian)

The equation

$$L(z)=z_{tt}+(2k+1)^2\pi^{-2}z_{xxxx}=\phi(x,t)+\mu f(z)=\psi(z),$$

subject to $z(0,t)=z_{xx}(0,t)=0$, $z(x,0)=z(x,1)$, $z(1,t)=z_{xx}(1,t)=0$, $z_t(x,0)=z_t(x,1)$, is considered. The author uses the method of successive approximations: $L(z_0)=0$, $L(z_{n+1})=\psi(z_n)$. Assuming appropriate conditions satisfied by ϕ and f, to ensure the existence of $z_1,\cdots,z_n,\cdots$, the author shows that these are uniformly bounded and equicontinuous functions in the domain $0\leqq x\leqq 1$, $0\leqq t\leqq 1$. He then invokes Arzela's theorem to show the existence of a convergent subsequence $\{\bar z_n\}$, $\bar z_n\rightarrow z$ as $n\rightarrow\infty$. It is then stated that $L(z)=\psi(z)$. No justification is given for this statement, and it would seem that the theorem of the paper is not proved. *R. Bellman* (Princeton, N. J.).

Zwirner, G. Sugli elementi uniti delle trasformazioni funzionali: alcune applicazioni al problema di Niccoletti per le equazioni differenziali a derivate parziali di tipo iperbolico. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 3, 44–49 (1947).

Étant donnée une fonction continue $f(x,y|z,\cdots,z_{pq},\cdots)$ $(p+q\leqq m+n-1)$ satisfaisant à la condition

$$(1)\qquad |f(x,y|0)|\leqq(1-K)N\qquad (K<1)$$

pour (x,y) dans le rectangle $a\leqq x\leqq b$, $c\leqq y\leqq d$, et satisfaisant à une condition de Lipschitz par rapport aux variables z_{pq}, avec le coefficient L satisfaisant à

$$L\leqq K\sum_p\sum_q(b-a)^{m-p}(d-c)^{n-q}/(m-p)!(n-q)!,$$

l'équation

$$\frac{\partial^{m+n}z}{\partial x^m\partial y^n}=f\left(x,y\,|\,z,\cdots,\frac{\partial^{p+q}z}{\partial x^p\partial y^q},\cdots\right)$$

admet une et une seule solution s'annulant sur un double système de $m+n$ caractéristiques appartenant au dit rectangle: $x=x_1,\cdots,x_m$, $y=y_1,\cdots,y_n$. Dans un deuxième théorème la proposition se trouve généralisée en supposant que les coefficients de Lipschitz L_{pq} pour chaque différence $|z_{pq}^1-z_{pq}^2|$ peuvent être différents, satisfaisant à la condition $\sum\sum_{pq}L_{pq}(b-a)^{m-p}(d-c)^{n-q}/(m-p)!(n-q)!<1$, et en substituant à la condition (1) l'autre condition plus précise que $|f(xy|z,\cdots,z_{pq},\cdots)|\leqq M$ pour

$$|z_{pq}|\leqq M(b-a)^{m-p}(d-c)^{n-q}/(m-p)!(n-q)!,$$

où M est une constante. La démonstration est fondée sur l'application d'un théorème fonctionnel de Hildebrandt et Graves et de Cacciopoli. *B. Levi* (Rosario).

Functional Analysis, Ergodic Theory

Arens, Richard. Duality in linear spaces. Duke Math. J. **14**, 787–794 (1947).

Soit X un espace vectoriel muni d'une topologie t_0 localement convexe (en abrégé: l.c.), X^* son dual (espace des formes linéaires continues dans X). L'auteur donne d'abord [th. 2] la condition pour qu'une topologie l.c. t sur X^* (et plus généralement sur un sous-espace total F de X^*) soit telle que X soit le dual de X^* (respectivement, F): il faut et il suffit que t soit plus fine que la topologie de la convergence simple dans X, et moins fine que la topologie de la convergence uniforme sur les parties convexes faiblement compactes de X; une telle topologie est dite réflexive. L'espace X^* étant muni d'une topologie réflexive t, et X identifié avec le dual de X^*, l'application du résultat précédent à X^* (au lieu de X) montre que t_0 est moins fine que la topologie κ de la convergence uniforme sur les parties convexes faiblement compactes de X^*; ceci donne les th. 3 et 4 de l'auteur, qui ne semble pas apercevoir leur parenté avec son th. 2: il ne ressort pas en effet clairement de ses notations que la notion d'ensemble faiblement compact dans un l.c. X ne dépend pas de la topologie de X, mais seulement du dual de X pour cette topologie (tout l'article gagnerait en clarté si dès le début on faisait jouer à X et F des rôles symétriques). Ces remarques montrent aussi que κ est la topologie l.c. la plus fine sur X pour laquelle X^* est le dual de X [th. 7], caractérisant ainsi cette topologie dont l'existence avait été démontrée par Mackey [Proc. Nat. Acad. Sci. U. S. A. **29**, 315–319 (1943); **30**, 24 (1944); ces Rev. **5**, 99; **8**, 708]. ~~[La bibliographie est incomplète, l'auteur attribuant par exemple à Alaoglu le théorème de N. Bourbaki sur la compacité faible de la boule unité dans le dual d'un espace normé [C. R. Acad. Sci. Paris **206**, 1701–1704 (1938)].~~]

J. Dieudonné (Nancy).

Segal, I. E. Postulates for general quantum mechanics. Ann. of Math. (2) **48**, 930–948 (1947).

A structure is defined, consisting of a (real) Banach space $\mathfrak{A}$ on which the set of all polynomials $P(x)$ in one variable, with real coefficients, operates so as to satisfy certain axioms; this is a modification of a set of axioms proposed by von Neumann [Rec. Math. [Mat. Sbornik] N.S. **1**(43), 415–482 (1936)]. Putting $f(x)=x^2/4$, and, for $U, V\epsilon\mathfrak{A}$, $U\circ V=f(U+V)-f(U-V)$, $\mathfrak{A}$ is called commutative if the (commutative) operation $U\circ V$ is distributive, associative, and satisfies $(\lambda U)\circ V=\lambda(U\circ V)$ for real λ; systems satisfying these conditions are identical with commutative Banach algebras, with unit-element, satisfying $\|U^2\|=\|U\|^2$ and $\|U^2-V^2\|\leqq\max(\|U\|^2, \|V\|^2)$; using the Gelfand-Mazur theorem and results of Stone and Gelfand, it is shown that such an algebra is isomorphic to the algebra of continuous functions over a compact space. As to general systems, the investigation is based on consideration of (real-valued) linear functions ω on $\mathfrak{A}$ (called "states") such that $\omega(U^2)\geqq 0$ for all U, and $\omega(1)=1$; extremal elements ("pure states") of the (convex) set of such functions are introduced by the Krein-Milman theorem; it is shown that there are enough "pure states" to separate any two elements of $\mathfrak{A}$. It is left undecided whether, under the author's axioms, every sum of squares must be a square; when that is the case, "pure states" of subsystems can be extended to "pure states" of the system.

A. Weil (Chicago, Ill.).

Akilov, G. P. On the extension of linear operations. Doklady Akad. Nauk SSSR (N.S.) **57**, 643–646 (1947). (Russian)

The author studies two problems on extension of linear operators from one normed space to another. A theorem of Phillips [quoted below] shows that these are both equivalent to the study of spaces X with the following property P_s, $s\geqq 1$: for every $Y\supseteq X$ there exists a projection of norm s of Y onto X. The author works with normed vector lattices of type B_1^+ (simultaneously of the types B_1 and K_s of Kantorovich [Rec. Math. [Mat. Sbornik] N.S. **7**(49), 209–284 (1940); these Rev. **2**, 317]). Theorem 1. If X is of type B_1^+, if the unit sphere of X has a least upper bound x_0, and if $\|x_0\|=s$, then X has the property P_s. A space X of type B_1^+ is called quasi-uniformly convex with modulus e if the conditions $\|x_1\|=\|x_2\|=1$ and $\inf(|x_1|, |x_2|)=0$ together imply $\|x_1+x_2\|\leqq 2-e$. Theorem 2. Let X be a reflexive, quasi-uniformly convex space of type B_1^+. If the unit sphere in X has no least upper bound or if the x_0 of theorem 1 satisfies $\|x_0\|>(2-e)^2/e$, then X does not have property P_1. Consequently the spaces L_p, l_p, and $l_{p,n}$ for $1<p<\infty$ and n sufficiently large do not have property P_1.

Reviewer's remarks. There is a typographical error in the definition of the characteristic ρ of a space; from the later theorems it is clear that $\rho=0$, not 1, when the unit sphere does not have a least upper bound. Since the space of bounded measurable functions on (0, 1) satisfies the hypotheses of theorem 1, theorem 1 is a proper extension of Phillips' result [Trans. Amer. Math. Soc. **48**, 516–541 (1940), cor. 7.2; these Rev. **2**, 318] that if m_T is the space of all bounded functions on a set T with least upper bound for norm, then m_T has property P_1.

M. M. Day.

Chen, Kien-Kwong. Weak convergence in hyperspace. Acad. Sinica Science Record **2**, 8–11 (1947).

The paper gives the usual proof of the well-known theorem that in the space L^p on k-dimensional Euclidean space the unit sphere is weakly compact.

T. H. Hildebrandt.

Mihlin, S. G. On the solution of linear equations in Hilbert space. Doklady Akad. Nauk SSSR (N.S.) **57**, 11–12 (1947). (Russian)

Let H be a Hilbert space and let A be a closed linear operator defined on a dense subspace D_A of H. Let A^* be the adjoint of A. Then $(A\phi, \psi)=0$ for all ϕ in D_A if and only if $A^*\psi=0$. Thus the range H' of A is dense in the orthogonal complement H_1^* of the null space H_0^* of A^*. In this note the author shows that a sufficient condition that $H'=H_1^*$ is that there exist a bounded linear operator M such that MA coincides on D_A with $I+T$, where I is the identity operator and T is completely continuous. The proof [which may be given in full in a few lines] consists in restricting A to the orthogonal complement of its null space and showing that the inverse of the result cannot be unbounded because of the complete continuity of T and the closedness of A. The author gives an example showing that the theorem need not be true if M is merely closed. [If A is any bounded operator with zero in its continuous spectrum the inverse of A will serve as an M for which T is zero.] He concludes with the remark that the theorem has as consequences certain theorems about the solution of singular integral equations.

G. W. Mackey.

Barsotti, I. A proof of two fundamental theorems on linear transformations in Hilbert space, without use of the axiom of choice. Bull. Amer. Math. Soc. 53, 943–949 (1947).

Without using the axiom of choice the author proves that (1) every closed linear manifold $\mathfrak{M}$ in the separable Hilbert space $\mathfrak{H}$ has an orthogonal complement $\mathfrak{H}\ominus\mathfrak{M}$, and (2) if H is a symmetric transformation whose domain is $\mathfrak{H}$, then H is bounded and therefore continuous. *R. A. Leibler.*

Krasnosel'skiĭ, M. A. On the deficiency numbers of closed operators. Doklady Akad. Nauk SSSR (N.S.) 56, 559–561 (1947). (Russian)

Let G be a connected open set of the complex λ-plane; let U_λ be a family of bounded linear operators with common domain $\mathfrak{L}$ in a Hilbert space, continuous for all λ in G with respect to the norm-topology for operators; let $U_\lambda x=0$ imply $x=0$; and let the range $\mathfrak{L}_\lambda=U_\lambda\mathfrak{L}$ be closed. By elementary continuity arguments it is then shown that the dimension of the orthogonal complement of $\mathfrak{L}_\lambda$ is constant over G. Now let A be a closed linear operator with domain $\mathfrak{D}(A)$; let R be the open set in the λ-plane where $\inf_x \|Ax-\lambda x\|/\|x\|>0$; and let $\mathfrak{L}_\lambda=(A-\lambda I)\mathfrak{D}(A)$. Application of the preceding result to the operator $U_\lambda=(A-\lambda I)(A-\lambda_0 I)^{-1}$, where λ and λ_0 are in R, shows that the dimension $d_\lambda(A)$ of the orthogonal complement of $\mathfrak{L}_\lambda$ is constant over each connected component of R. The number $d_\lambda(A)$ is termed the deficiency-number of A at λ. When A is Hermitian the well-known facts concerning its deficiency-index appear as a corollary of the above result. *M. H. Stone* (Chicago, Ill.).

Krasnosel'skiĭ, M., and Kreĭn, S. On the center of a general dynamical system. Doklady Akad. Nauk SSSR (N.S.) 58, 9–11 (1947). (Russian)

G. D. Birkhoff has shown [cf. Dynamical Systems, Amer. Math. Soc. Colloquium Publ., v. 9, New York, 1927, p. 196] that for an ordinary dynamical system (a one-parameter transformation group) the relative sojourn of an arbitrary point of the phase space in a neighborhood of the center is 1. The present paper extends this result to a more general transformation group.

Let G be a connected locally compact multiplicative topological group with identity e. Suppose also that G is not compact. Let M be a compact metric space and let G act as a transformation group on M; that is to say, to $g\epsilon G$ and $x\epsilon M$ there is assigned $g(x)\epsilon M$ such that (1) $e(x)=x$ $(x\epsilon M)$; (2) $g'(g(x))=(g'g)(x)$ $(g', g\epsilon G; x\epsilon M)$; (3) the function $g(x)$ maps $G\times M$ continuously into M. A point $x\epsilon M$ is said to be nonwandering provided that to each neighborhood U of x and each compact set Q in G there corresponds $g\epsilon G-Q$ such that $U\cap g(U)\neq 0$. Let M_1 be the set of nonwandering points of M. The authors remark that if V is a neighborhood of M_1 and if $x\epsilon M$, then there exists a compact set Q in G such that $g(x)\epsilon V$ for all $g\epsilon G-Q$. [Cf. Birkhoff, loc. cit., p. 193.] For $x\epsilon M$, $N\subset M$ and $H\subset G$ let $\{x, N; H\}$ denote the set of all $g\epsilon H$ for which $g(x)\epsilon N$. Let m be a right invariant Haar measure in G. It is pointed out that, if V is an open neighborhood of M_1, then $m\{x, M-V; G\}$ is bounded uniformly for $x\epsilon M$. [Cf. Birkhoff, loc. cit., p. 193.] Define a transfinite sequence $M\supset M_1\supset\cdots\supset M_\alpha\supset M_{\alpha+1}\supset\cdots$ of nonvacuous closed invariant sets in M as follows: $M_{\alpha+1}$ is the set of nonwandering points relative to the space M_α; $M_\alpha=\bigcap_{\beta<\alpha}M_\beta$ in case α is a limit ordinal. The smallest set Z in this sequence is called the center. The following theorem is proved. If V is an open neighborhood in M of the center Z and if W is an open neighborhood of e in G such that $\bar{W}$ is compact, then $\lim_{n\to+\infty} m\{x, V; W^n\}(mW^n)^{-1}=1$ uniformly for $x\epsilon M$.

W. H. Gottschalk (Princeton, N. J.).

Calculus of Variations

***Morse, Marston. Functions on a metric space and a setting for isoperimetric problems.** Studies and Essays Presented to R. Courant on his 60th Birthday, January 8, 1948, pp. 253–263. Interscience Publishers, Inc., New York, 1948. $5.50.

Let F and H be lower semicontinuous functions defined on a metric space S and let F be positive. In classical isoperimetric problems one seeks a minimum of H subject to the condition $F=1$. This paper outlines in considerable detail a new approach to such problems. Instead of restricting attention to the behavior of H on the subset S^* of S for which $F=1$, the author brings into consideration the properties of H on the subset S^F of S for which $F\leqq 1$. This extension permits the derivation of critical point relations between the homotopic critical (in a sense defined in the paper) points of H relative to S^F and the Betti numbers of S^F. These critical points of H divide into two classes, those on the subset of S for which $F<1$ and those on S^*, which include the solutions of the isoperimetric problem. The power of the method lies in the fact that the problem of determining Betti numbers and the critical points for which $F<1$ is likely to be a much simpler problem than that of finding the solutions of the isoperimetric problem and from knowledge of the first two it is possible to derive information concerning the last. The theory is discussed in two cases: (a) S is an m-dimensional manifold and (b) S is a compact metric space. *G. A. Hedlund.*

Damköhler, Wilhelm, und Hopf, Eberhard. Über einige Eigenschaften von Kurvenintegralen und über die Äquivalenz von indefiniten mit definiten Variationsproblemen. Math. Ann. 120, 12–20 (1947).

In this paper are given comparatively simple proofs of sufficient conditions under which a given integrand differs by an exact differential from an integrand which is nonnegative almost everywhere. In the simplest case the integrand is assumed to be locally bounded above (not necessarily continuous) and the value of the integral is assumed to be bounded below for all paths joining two fixed points. Extensions are obtained to the case when admissible line elements are required to satisfy differential equations (problem of Lagrange) or differential inequalities. In this case the integrand obtained is shown to be nonnegative for almost all points and for all admissible line elements.

L. M. Graves (Bloomington, Ind.).

Viola, Tullio. Procedimenti costruttivi per le estremanti di un funzionale. Rend. Circ. Mat. Palermo 62, 105–136 (1939).

The author considers an integral $J=\int f(x, y, y')dx$ and a class E of curves $y=y(x)$ in which the derivatives $y'(x)$ satisfy a uniform Lipschitz condition. In the topology determined by the norm $\|y\|=\max\{|y(x)|, |y'(x)|\}$, the integral J is continuous, and bounded subsets of E are compact. Consequently if we consider a subclass A of E

consisting of all $y(x)$ for which y and possibly y' are required to take prescribed values at certain prescribed points $x=a_i$, and if A is nonnull, then J will have a maximum and a minimum on A, since A is certainly compact and closed. For the special case of the length integral, it is shown that maximizing and minimizing curves can be constructed lying in certain special subclasses of E which are dense on E.

L. M. Graves (Bloomington, Ind.).

Ewing, George M. Variation problems formulated in terms of the Weierstrass integral. Duke Math. J. 14, 675–687 (1947).

This paper gives proofs of the main existence and semicontinuity theorems for the curve-functions $J(C)$ of the calculus of variations defined as Weierstrass integrals $\int f(x, dx)$. Of methodological interest is the consideration of the integral $\int f(y, dx)$ related to two curves C_1: $x=x(t)$, C_2: $y=y(t)$, $a\leqq t\leqq b$. The Tonelli-Aronszajn theorem [cf. Aronszajn, Revue Sci. 78, 233–239 (1940); these Rev. 1, 305] on the existence of $\int f(x, dx)$ when the curve C is rectifiable, $f(x, r)$ continuous in (x, r) and positively homogeneous in r, is extended to the integral $\int f(y, dx)$, in which C_1 is only assumed to be rectifiable. Following an idea of Tonelli for the case of the plane, the author replaces f by an ϵ-approximation f^* with continuous second partial derivatives (regularization) and then substitutes for f^* an integrand $f^*+k|r|$ with sufficiently large k to be convex (convexification).

The last paragraph of the paper is devoted to a simplification of a proof of an upper reducibility property given in a previous paper [G. Ewing and M. Morse, Ann. of Math. (2) 44, 339–353 (1943); these Rev. 5, 270] by using the reduced Morse parametrization and the Weierstrass integral.

C. Y. Pauc (Cape Town).

TOPOLOGY

Sierpinski, W. Un théorème sur les espaces métriques denses en soi. Proc. Benares Math. Soc. (N.S.) 7, no. 2, 29–31 (1945).

It is proved that any metric space without isolated points is "resolvable," i.e., expressible as the union of two disjoint dense subsets. A more general theorem was proved by E. Hewitt [Duke Math. J. 10, 309–333 (1943); these Rev. 5, 46]. [Hewitt proves also that some completely regular spaces are irresolvable.] *R. Arens* (Los Angeles, Calif.).

Krishnan, V. S. A weak homeomorphism between topological spaces and a characterization of completely regular spaces. J. Indian Math. Soc. (N.S.) 10, 37–56 (1946).

The paper deals with T- (but not necessarily T_0-)spaces, the theory being trivial or known if applied only to T_0-spaces. A many-many mapping f is called continuous if f^{-1} is open and preserves complements; f is a weak homeomorphism if f and f^{-1} are both continuous. Thus a T-space is weakly homeomorphic to precisely one T_0-space; a space is completely regular if and only if it is weakly homeomorphic to a subset of a product of quasi- (i.e., pseudo-) metric spaces. *R. Arens* (Los Angeles, Calif.).

de Groot, J. Topological characterization of all subsets of the real number system. Nederl. Akad. Wetensch., Proc. 50, 876–884 = Indagationes Math. 9, 387–395 (1947).

A metric separable space X is homeomorphic with a subset of the space of real numbers if (a) quasicomponents of X are either points of open, closed or half-open arcs, (b) no interior point of an arc in X is a limit point of the complement and (c) every point of the space has "small" neighborhoods with compact boundaries. This generalizes in a certain direction results of R. L. Moore and J. R. Kline [Ann. of Math. (2) 20, 218–223 (1919)]. See also L. Zippin [Trans. Amer. Math. Soc. 34, 705–721 (1932)].

J. L. Kelley (Berkeley, Calif.).

Doss, Raouf. On continuous functions in uniform spaces. Ann. of Math. (2) 48, 843–844 (1947).

The author proves that a completely regular T_0-space admits unbounded real-valued continuous functions if and only if it admits a uniform structure the completion of which is not compact. *E. Hewitt* (Chicago, Ill.).

Pontryagin, L. S. Characteristic cycles on differentiable manifolds. Mat. Sbornik N.S. 21(63), 233–284 (1947). (Russian)

This is a detailed account of results already noted [C. R. (Doklady) Acad. Sci. URSS (N.S.) 35, 34–37 (1942); these Rev. 4, 147; some necessary definitions will be repeated here for the reader's convenience]. The author remarks on the close connection of his work to that of Whitney [see the cited review]. In a paper to be published soon, he promises a closer study of the relation of his more general results to those of Stiefel, as well as an alternative procedure for the actual calculation of the characteristic cycles with which this paper is concerned. He regards as still open the question of the "newness" of invariants of this type.

Let M^k be a differentiable orientable manifold of dimension k and $f(M^k)$ a homeomorphic image, with continuously turning tangent, in a Euclidean R^{k+l}. Let $H(K, l)$ be the space of oriented k-planes through the origin of R^{k+l}; this is of dimension $k\cdot l$. For $x\epsilon M^k$, let $T(x)$ denote the element of $H(k, l)$ which is parallel to the tangent k-plane to $f(M^k)$ at $f(x)$. The map $T(x)$ is called the tangential map of the abstract M^k and is independent, to within a homotopy, of the particular imbedding $f(M^k)$. If Z is a cycle of $H(k, l)$ of dimension $kl-r$, $r\leqq k$, there exists a homeomorphism T_1 of M^k into $H(k, l)$ which approximates T and for which Z and $T_1(M^k)$ are in general position. Let X denote the inverse image under T_1 of the intersection $Z\times T_1(M^k)$. The homology class of the $(k-r)$-dimensional cycle X depends on Z (and, of course, M^k) alone. When Z is one of a set of base cycles of $H(k, l)$, the associated X is called a generating characteristic cycle of M^k.

The author selects his base cycles following procedures credited to Ehresmann [reference here to Ann. of Math. (2) 35, 396–443 (1934); J. Math. Pures Appl. (9) 16, 69–100 (1937)]. Let $\omega(i)$, $i=1, \cdots, k$, be an integral-valued nondecreasing function with $0\leqq\omega(i)\leqq l$. Let $R_1\subset R_2\subset\cdots\subset R_k$ be some associated monotonic system of linear spaces in R^{k+l} and let the dimension of R_i be $\omega(i)+i$. Denote by $Z(\omega)$ the set of all $R^k\epsilon H(k, l)$ such that the dimension of the intersection of R^k and R_i is not less than i. In general, $Z(\omega)$ is a closed pseudomanifold, and is of dimension $r(\omega)=\sum\omega(i)$. When it is orientable, $Z(\omega)$ may be regarded as a cycle with integral coefficients; when nonorientable, as a cycle mod 2. In the latter case, if Y is a nonorientable cycle, the author associates with it a cycle with integral

coefficients, designated by ΓY, whose homology class is determined by Y. When $Z(\omega)$ is of dimension zero it is an integer or an "integer mod 2," but still regarded as a characteristic cycle. In all cases, $Z(\omega)$ is determined by the function ω, not the particular associated sequence R_i. The author introduces a complementary function $\chi=\chi(i)$, defined by $\omega+\chi\equiv l$; and finds his base cycles among those whose functions satisfy $l-1\geqq\chi(i)\geqq 0$ and, also, $r(\chi)=\sum\chi(i)\leqq k$. As a change in notation, $Z(\omega)\equiv Z_\chi$ for any ω and complementary χ.

Certain classes of functions χ are now investigated. Let $i_1, i_2, \cdots, i_{n-1}$ designate the "jump-spots" for the function: thus, $\chi(i_{m+1})<\chi(i_m)$; and let $i_0=0$, $i_n=k$. Define $a_k=i_k-i_{k-1}$, $b_k=\chi(i_k)-\chi(i_{k-1})$, and $b_n=\chi(k)$. Now associate with χ the sequence $a_1, b_1, a_2, \cdots, a_{n-1}, b_{n-1}, a_n$, or the similar sequence $a_1, b_1, \cdots, b_{n-1}$, according as $b_n>0$ or $b_n=0$. Consider the class of functions χ, designated by X', for which $a_1\geqq 2$. This has a subclass X (not to be confused with the symbol for the characteristic cycles, which hereafter have subscripts) whose associated sequence, above, consists of even numbers. Functions in X' but not in X belong to a class X_b if the first odd integer in the appropriate sequence is a "b." These two classes suffice for theorem 1 which is as follows. A canonical basis for homologies of dimension $kl-r$, $r\leqq l-1$, in the manifold $H(k, l)$ may be built up from cycles (i) Z_χ, $\chi\epsilon X$, $r(\chi)=r$; (ii) $Z_{\chi'}$, $\chi'\epsilon X_b$, $r(\chi')=r-1$. The cycles of the first kind are free, those of the second are of order two.

Of the associated characteristic cycles X_χ, the zero dimensional X_χ, $\chi\epsilon X_b$, are identically zero. The particular cycle, $X_1=X_\chi$ with $\chi\equiv 1$, is zero when k is odd. It is to be proved in the forthcoming paper, referred to above, that in the case that k is even X_1 is the Euler characteristic of M^k.

Theorem 2. Let $-M^k$ denote the manifold which coincides with M^k geometrically, but is oppositely oriented. Then (i) $X_1(-M^k)\sim X(M^k)$; (ii) $X_\chi(-M^k)\sim -X_\chi(M^k)$, if $\chi\epsilon X$ but $\chi\not\equiv 1$; (iii) $X_\chi(-M^k)\sim X_\chi(M^k)$ in all cases not under (i) and (ii). This theorem gives a criterion that a manifold be asymmetric, that is, that it shall not admit a homeomorphism on itself with reversal of orientation. The final principal theorem is as follows. Theorem 3. If an orientable closed manifold M^k can serve as the boundary of an orientable bounded manifold M^{k+1}, then all characteristic cycles of the manifold M^k are equal to zero, with the possible exception of X_1 which is then necessarily even. To this theorem the author remarks that it would be of some consequence if the known result that every M^3 can serve as boundary could be generalized to M^5.

The four-dimensional manifolds M^4 have three generating characteristic cycles. The first, X_1, gives the Euler characteristic. The second, denoted by X_{22}, is also zero-dimensional, and is determined by the function $\chi(1)=\chi(2)=2$, $\chi(3)=\chi(4)=0$. These two cycles are equal modulo two, but are essentially independent [as is to be shown in the forthcoming paper]; X_{22} is not contained among the cycles of Stiefel. The third cycle, X_{11}, is two-dimensional, and of order two, defined by $\chi(1)=\chi(2)=1$, $\chi(3)=\chi(4)=0$. Here the author observes that the problem of classifying the maps of S^{n+3} on S^n could be much advanced by a fuller knowledge of the relations existing between the three cycles; for example, it would be important to know whether $X_{11}\sim 0$ implies that $X_{22}\sim 0$.

All the statements of this review will be found in the very substantial introduction. The body of the paper contains six sections. The first is devoted to geometrical properties of the manifolds $H(k, l)$; for example, it is shown that, for $kl>1$, $H(k, l)$ is simply connected, and for $k\geqq 2$, $l\geqq 2$, $H(k, l)-Z(\omega)$ is simply connected. In § 2 there is defined a suitable orientation for the orientable $Z(\omega)$; § 3 is devoted to the tangential mapping and characteristic cycles; § 4 is entitled "the cellular subdivision of $H(k, l)$"; § 5 is devoted to the homology basis for $H(k, l)$. The final section is entitled "certain properties of characteristic cycles" and contains the proofs of theorems 2 and 3. *L. Zippin.*

Eckmann, Beno. On complexes over a ring and restricted cohomology groups. Proc. Nat. Acad. Sci. U. S. A. **33**, 275–281 (1947).

Let R be a ring with a unit. An R-complex $\mathfrak{C}$ is a sequence of R-modules $C_{-1}, C_0, C_1, \cdots$ and R-homomorphisms $\partial: C_n\rightarrow C_{n-1}$ such that $\partial\partial=0$ and that C_n is R-free for $n\geqq 0$. Let J be any Abelian coefficient group and ψ a group of additive homomorphisms $R\rightarrow J$. A cochain $f^n: C_n\rightarrow J$ is called a ψ-cochain if, for each $a\epsilon C_n$, $f^n(ra)$ treated as a function of $r\epsilon R$ is in ψ. Using ψ-cochains throughout the cohomology group H^n_ψ is defined. If J is an R-module and ψ is the group of all R-homomorphisms $R\rightarrow J$, H^n_ψ is the equivariant cohomology group as defined by the reviewer [Trans. Amer. Math. Soc. **61**, 378–417 (1947); these Rev. **9**, 52]. The author exhibits other interesting examples of ψ. The main result is: if the ordinary homology groups of $\mathfrak{C}$ are trivial for all dimensions less than N, then R, C_{-1}, J and ψ determine the groups H^n_ψ for $n<N$ as well as a suitable subgroup of H^N_ψ. *S. Eilenberg* (New York, N. Y.).

White, Paul A. On the equivalence between avoidability and co-local connectedness. Anais Acad. Brasil. Ci. **19**, 143–151 (1947).

A proof that two definitions of a generalized manifold, given by Wilder and the reviewer, are equivalent.
E. G. Begle (New Haven, Conn.).

GEOMETRY

★Bernays, Paul. Bemerkungen zu den Grundlagen der Geometrie. Studies and Essays Presented to R. Courant on his 60th Birthday, January 8, 1948, pp. 29–44. Interscience Publishers, Inc., New York, 1948. $5.50.

The first of the two notes forming this paper has its origin in Hilbert's investigation of the role played in axiomatic geometry by the theorem that the base angles of an isosceles triangle are equal [Grundlagen der Geometrie, 7th ed., 1930; see appendix 2, p. 134]. This theorem is not provable in a system obtained from Hilbert's by omitting the (spatial) incidence axioms I, 4–8, the continuity axioms V, 1–2, and restricting the triangle congruence axiom III, 5 to triangles with the same orientation. The question arises whether the assumption of this theorem as an additional axiom implies the unrestricted congruence axiom III, 5. This seems not to be the case, but the author shows that if besides adjoining the base angle theorem it is also assumed that two angles with the same vertex are not congruent if the sides of one lie in the interior of the other, then the unrestricted congruence axiom III, 5 may be proved.

The second note concerns those operations which are possible with a straight edge with two marks on it. The problems of angle trisection and cube duplication, for example, are solvable. It is shown that the field of con-

structible numbers consists of those numbers which, starting from 1 (the distance between the two marks) are generated by the rational operations, the real radicals $\sqrt{c}$ and $\sqrt[3]{c}$, $c>0$, as well as the positive roots of the trisection equation $x^3-3x=2c$, $0<c<1$. *L. M. Blumenthal.*

✶Malengreau, Julien. Considérations sur les Fondements de la Mathématique. F. Rouge & Cie., S. A., Lausanne, 1947. 39 pp. 6 Swiss francs.

The author makes some general remarks upon desirable features of a postulational basis for geometry (e.g., compatibility, independence, completeness).

L. M. Blumenthal (Columbia, Mo.).

Forder, H. G. The cross and the foundations of Euclidean geometry. Math. Gaz. **31**, 227–233 (1947).

The paper sketches a foundation of Euclidean geometry in which the notions of "cross" and "signed angle" are basic. A cross refers to two lines instead of two rays, and is taken mod 2 right angles. The author states that a large part of Euclidean geometry can be developed as consequences of the five postulates he gives in which "cross" is the only primitive notion. The notion is, however, inadequate for the introduction of order, for which "signed angle" is used. An appendix considers some of the consequences of attaching to each three points P, Q, R a real (signed) number PQR (interpretable as the signed area of the triangle with vertices P, Q, R) such that, for each four points A, B, C, D, the relation $ABC-BCD+CDA-DAB=0$ is satisfied.

L. M. Blumenthal (Columbia, Mo.).

✶Bachmann, F. Konstruierbarkeit mit Lineal, Rechtwinkelmass und Eichmass in einer Geometrie mit Euklidischer Metrik, ohne Voraussetzung des Parallelenaxioms. Ber. Math.-Tagung Tübingen 1946, pp. 38–40 (1947).

Summary of a lecture on constructions in a plane geometry satisfying Hilbert's axioms of connection, order and congruence, the axiom of parallels being omitted. The instruments used are a straight edge for the construction of the straight line joining any two given points, a unit measure which serves to lay off a segment of unit length on a given line from a given point, and a set square to construct the perpendicular to a given line through a given point. *F. A. Behrend* (Melbourne).

✶Flanders, Donald A. Angles between flat subspaces of a real n-dimensional Euclidean space. Studies and Essays Presented to R. Courant on his 60th Birthday, January 8, 1948, pp. 129–138. Interscience Publishers, Inc., New York, 1948. $5.50.

The determination of the mutual orientation of two flat subspaces in Euclidean n-dimensional space was first given by C. Jordan [Bull. Soc. Math. France **3**, 103–174 (1875)]. The author proposes "to treat this problem once more in somewhat more modern style." After a short discussion of some fundamental notions of n-dimensional geometry (sum, intersection of flat subspaces, projection) in terms of matrix notation he proceeds to investigate the angles between two flat subspaces S_n^p and S_n^q. Here S_n^p is a p-dimensional flat subspace given by $x=At$, i.e.,

$$\begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} = \begin{bmatrix} a_1^1 & \cdots & a_1^p \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ a_n^1 & \cdots & a_n^p \end{bmatrix} \cdot \begin{bmatrix} t_1 \\ \cdot \\ \cdot \\ \cdot \\ t_p \end{bmatrix}, \qquad AA'=E,$$

(A' the transpose of A, E the unit matrix) and S_n^q a q-dimensional subspace given by $x=Bt$ ($BB'=E$). The squares of the cosines of the angles will be furnished by the characteristic values of the matrix $B'AA'B$. We get in this way a set of plane angles, whose number is min (p, q), of which (in case $p+q-n>0$) $p+q-n$ are 0, corresponding to the fact that the subspaces intersect in a $(p+q-n)$-dimensional subspace. *E. Egerváry* (Budapest).

Yang, Chung-Tao. Certain chains in a finite projective geometry. Acad. Sinica Science Record **2**, 44–46 (1947).

A chain is defined as the set of p^r+1 points on a line of a $PG(2, p^n)$ contained in a subplane $PG(2, p^r)$. This follows the usage of Veblen and Bussey [Trans. Amer. Math. Soc. **7**, 241–259 (1906)]. A simple chain ($r=1$) is a Möbius net and is determined by any three points of a line. Thus the chains on a line have combinatorial properties analogous to circles. The author states without proof several theorems about chains with $r=2^m$ contained in chains with $r=2^{m+1}$.

M. Hall, Jr. (Columbus, Ohio).

Goodstein, R. L. Commutative involutions. Math. Gaz **31**, 224–226 (1947).

The author studies involutions of points on lines or conics. Express by $x\mathfrak{J}y$ the statement that the point y corresponds to the point x in an involution $\mathfrak{J}$. Let $\mathfrak{J}$ and $\mathfrak{J}'$ be involutions on the same line or conic. If $x\mathfrak{J}y$ and $y\mathfrak{J}'y'$ then a projectivity $x\mathfrak{J}\mathfrak{J}'y'$ (or $y'\mathfrak{J}'\mathfrak{J}x$) is established on the line or conic. The involutions $\mathfrak{J}$ and $\mathfrak{J}'$ are said to be commutative if $\mathfrak{J}\mathfrak{J}'=\mathfrak{J}'\mathfrak{J}$. This is the case if and only if $\mathfrak{J}\mathfrak{J}'$ is also an involution. Various properties of commutative involutions are discussed. Examples are as follows. (1) The family of conics through four points of a plane determines an involution on a fixed line l. It is shown that this involution is commutative with the involution of conjugate points on l with respect to any conic of the family. (2) If any two collinear involutions are given, there exists a third involution, collinear and commutative with both.

E. Lukacs (Cincinnati, Ohio).

Kollros, Louis. Solution d'un problème de Steiner. Elemente der Math. **2**, 105–107 (1947).

The following problem is considered: to place two given coplanar projective flat pencils S, S' so that they shall generate either an ellipse which is the closest to a circle or a hyperbola which is the farthest from an equilateral hyperbola. If the pencil S is kept fixed and the pencil S' is revolved about its vertex S', it will in one of its positions be perspective to S. Two cases are considered: the points S, S' lie on the same side or on opposite sides of the axis of perspectivity of the two pencils. By synthetic reasoning the author shows that in the first case the pencil of conics generated by the fixed pencil S and the revolving pencil S' includes one and only one ellipse e_M in which the angle formed by the equal conjugate diameters is maximum, while in the second case the pencil of conics includes one and only one hyperbola h_m whose asymptotes form a minimum angle. The position of the pencil S' giving rise either to the ellipse e_M or to the hyperbola h_m is determined by a simple construction and together with the pencil S constitutes the solution of the problem. *N. A. Court.*

Grace, J. H. A question concerning Aronhold's theorems on bitangents. Proc. Edinburgh Math. Soc. (2) 8, 37–38 (1947).

"The gist of Aronhold's results [on the bitangents to nonsingular quartic curves] can be stated in a very simple

form, namely: given seven lines in a plane it is possible to derive from them uniquely and symmetrically a quartic curve that has them for bitangents." The author proves by simple invariant theory that there does not exist an analogous theorem in space; more generally that a sextic space curve of genus four cannot stand in unique, symmetric and projective relation to eight given planes.
R. A. Johnson (Brooklyn, N. Y.).

Lippert, V. Figures formées par les pôles d'un plan et les polaires d'un point, des points d'une droite et des points d'un plan par rapport aux coniques de la surface de Steiner (surface romaine). Acta Fac. Nat. Univ. Carol., Prague no. **172** (1939), 21–27 (1946). (Czech and French)

A list of special properties of the Steiner surface, starting from Lie's theorem. No proofs are given.
J. G. Semple (London).

Droussent, Lucien. On a theorem of J. Griffiths. Amer. Math. Monthly **54**, 538–540 (1947).

The author deals with the cyclic quadrangle and its "anticenter," the point of intersection of the four nine-point circles and the four Simson lines. All the results are to be found in available works or are easily deducible therefrom. [See, e.g., Johnson, Modern Geometry, Houghton Mifflin, Boston, 1929 (cited by the translator), §§ 265, 315, 327, 415.] *R. A. Johnson* (Brooklyn, N. Y.).

Venkataraman, M. A new proof of the centre circle chain. J. Indian Math. Soc. (N.S.) **10**, 65–67 (1946).

By the "centre-circle theorem" the author means the theorem associated with the names of de Longchamps and Pesci [cf. Coolidge, A Treatise on the Circle and the Sphere, Oxford, 1916, p. 92]. The theorem is as follows. The centre-circle of three coplanar lines is the circumcircle of their triangle. Given n coplanar lines, the centre-circles of sets of $n-1$ of them will be concurrent, and their centers will lie on a circle, the centre-circle of the n lines. The author cites proofs of this theorem by Morley, Grace, White, Iyer and Bhimasenarao, using a surprising diversity of methods. He offers a new proof based on methods of higher plane curves. *R. A. Johnson* (Brooklyn, N. Y.).

Thébault, V. Sur une nouvelle sphère du tétraèdre orthocentrique. Bull. Soc. Math. France **74**, 26–30 (1946).

Thébault, V. Sur le premier point de Lemoine du tétraèdre. Ann. Soc. Sci. Bruxelles. Sér. I. **61**, 215–221 (1947).

Thébault, Victor. Sur une sphère associée au tétraèdre. C. R. Acad. Sci. Paris **225**, 1260–1261 (1947).

Convex Domains, Integral Geometry

de Rham, Georges. Un peu de mathématiques à propos d'une courbe plane. Elemente der Math. **2**, 73–76, 89–97 (1947).

For a convex polygon P in the plane denote by $P'(P)$ the polygon whose vertices are the points which divide the sides of P into 3 equal parts (if P has n vertices the $P'(P)$ has $2n$ vertices). If P_0 is a square and $P_n=P'(P_{n-1})$, then P_n tends to an interesting convex curve C. For instance, C has everywhere a tangent, but every subarc of C contains both points with vanishing and infinite radius of curvature.
H. Busemann (Los Angeles, Calif.).

Day, Mahlon M. Polygons circumscribed about closed convex curves. Trans. Amer. Math. Soc. **62**, 315–319 (1947).

The following theorem is proved by means of a slightly generalized form of the Poincaré fixed point theorem for the ring (not formulated in the paper). Given a plane convex closed curve C with a center and positive numbers $\lambda_1, \cdots, \lambda_n$ with product 1. Then there exists a $(2n)$-sided convex polygon with the same center circumscribed about C such that the sides of the ith pair of opposite sides are divided in the ratio λ_i by their points of contact with C. A similar theorem for curves without center and arbitrary circumscribed n-sided polygons holds. In the case $\lambda_1=\cdots=\lambda_n=1$ both theorems are proved in a second way based on the fact that a circumscribed n-sided polygon with minimal area has its points of contact at the midpoints of the sides. This method gives analogous theorems for k-dimensional spaces, where the midpoints of the sides are replaced by the centroids of the faces of circumscribed polyhedra. Some special cases and this second method are well known. Applications to the theory of normed linear spaces are given in another paper of the author [same vol., 320–327 (1947); these Rev. **9**, 192]. *W. Fenchel* (Copenhagen).

Vincze, Stephan. Über den Minimalkreisring einer Eilinie. Acta Univ. Szeged. Sect. Sci. Math. **11**, 133–138 (1947).

The paper contains: a proof for the fact that the minimal concentric ring containing a given plane convex curve K introduced by Bonnesen is unique [concerning this part of the paper see T. Bonnesen, C. R. Acad. Sci. Paris **188**, 35–37 (1929) or Mat. Tidsskr. B. **1928**, 38–77, where similar considerations are used in a general study of linear approximations, including the problem of the minimal ring]; inequalities between the radii of the ring and of the circumscribed and inscribed circles; convexity properties of mean values of the distance PQ between a fixed point P inside K and a variable point Q on K. *W. Fenchel* (Copenhagen).

Valentine, F. A. The determination of connected linear sections. Duke Math. J. **14**, 723–730 (1947).

The starting point of the paper is the following remark. Let P be a property of hyperplanes in R_n $(n\geqq 2)$ and K the set of all points x in R_n such that each hyperplane through x has property P. Then each component of K is convex. This is applied in the following case: S is a given continuum in R_n; a hyperplane has property P if it intersects S in a connected (or empty) set. If $n=2$ and "empty" is omitted, this gives a theorem of Brunn on star-like sets. The author derives a necessary and sufficient condition that a component of K be closed and proves among other things that a hyperplane of support to a component of K can intersect S in at most three components. More detailed results, also concerning the number of components of K, are obtained in the case $n=2$. *W. Fenchel* (Copenhagen).

***Stoker, J. J. Open convex surfaces which are rigid.** Studies and Essays Presented to R. Courant on his 60th Birthday, January 8, 1948, pp. 407–420. Interscience Publishers, Inc., New York, 1948. $5.50.

The rigidity of two classes of open convex surfaces is proved. (1) The surface is given in the form $z=z(x, y)$, where the function $z(x, y)$ is defined in the whole (x, y)-plane and $\lim z=\infty$ as $x^2+y^2\rightarrow\infty$. (2) The surface is obtained by revolving a curve $z=z(r)$ about the z-axis. The function $z(r)$ is defined in an interval $0=a\leqq r<b$ with $z_r=0$ for $r=a$ and $\lim_{r\to b} z(r)=\infty$. The functions $z(x, y)$ and $z(r)$ are

assumed to be single-valued and three times continuously differentiable. (3) If we put $a>0$ in (2), we obtain surfaces of revolution with circular holes. They too are proved to be rigid, though in a somewhat restricted sense only. The proofs of (1) and (2) make use of a theorem of S. Bernstein on bounded solutions of certain elliptic differential equations [Math. Z. **26**, 551–558 (1927)]. The third result is obtained by showing that a certain improper double integral whose integrand is nonnegative vanishes [cf. Blaschke, Differentialgeometrie I, 3d ed., Berlin, 1930, § 93].

P. Scherk (Saskatoon, Sask.).

Hadwiger, H. Ueber das Volumen der Parallelmengen. Mitt. Naturforsch. Ges. Bern (N.F.) **3**, 121–125 (1946).

Let A be a closed bounded domain in E^n whose boundary B has continuous principal curvatures. Generalizing a concept which he introduced for curves [Comment. Math. Helv. **18**, 59–72 (1945); these Rev. **7**, 260] the author calls A "unterkonvex" of degree α if α is the least upper bound of those ρ for which the intersection of any sphere of radius ρ with A has Euler characteristic 1, and "überkonvex" of degree β if the closure of the complement of A is unterkonvex of degree β. Then the volume V_ρ of the parallel set to A at distance ρ satisfies for $-\beta\leqq\rho\leqq\alpha$ the relation $V_\rho=V_0+c_1\rho+\cdots+c_n\rho^n$, where c_1 is the area of B and generally

$$c_{k+1}=n^{-1}\binom{n}{k+1}\binom{n-1}{k}^{-1}\int_B\sum k_{\lambda_1}\cdots k_{\lambda_k},$$

where the k_i are the principal curvatures of B and the summation is to be extended over all combinations $(\lambda_1,\cdots,\lambda_k)$ of order k of $1, 2, \cdots, n-1$.

H. Busemann.

Fejes Tóth, L., und Hadwiger, H. Mittlere Trefferzahlen und geometrische Wahrscheinlichkeiten. Experientia **3**, 366–369 (1947).

[In the original the first author's initial was misprinted as A.] Let $\{G_\nu\}$, $\nu=1, 2, \cdots$ be a system of simply connected domains in the plane with uniformly bounded diameters, areas F_ν and boundary lengths $L_\nu<\infty$. Denote by $N(R)$ the number of G_ν which lie in the circle C_R of radius R about the origin, and for $R\rightarrow\infty$ let the limits $D_0=\lim N(R)\pi^{-1}R^{-2}$, $\bar F=\lim\sum F_\nu N^{-1}(R)$ and $\bar L=\lim\sum L_\nu N^{-1}(R)$ exist, where the summation is extended over the ν for which $G_\nu\subset C_R$. For another simply connected domain G of area F and boundary length L let S denote the sum over ν of the number of components of GG_ν. Then a suitably defined mean value $\bar S$ of S, if G is moved in the plane, exists and (*) $\bar S=D_0(F+\bar F+L\bar L(2\pi)^{-1})$. The proof of (*) is postponed to another paper. The present note gives various applications of (*). If the G_ν are points then $\bar F=\bar L=0$ and (*) yields D_0F for the average number of points G_ν in C_R. If the G_ν are segments (*) leads to a solution of Buffon's needle problem. If $\{G_\nu\}$ constitutes a paving of the plane with congruent figures then $D_0=F_\nu^{-1}$ and (*) yields various known formulas for special pavings.

H. Busemann.

Algebraic Geometry

Hodge, W. V. D. Harmonic integrals on algebraic varieties. Proc. Cambridge Philos. Soc. **44**, 37–42 (1948).

The author discusses the results in chapter IV of his book [The Theory and Applications of Harmonic Integrals, Cambridge University Press, 1941; these Rev. **2**, 296], where he uses a metric of Kähler type, i.e., a ds^2 which everywhere locally is of the form

$$ds^2=\sum\frac{\partial^2\psi}{\partial z_i\partial\bar z_j}dz_id\bar z_j.$$

He first remarks that his results depend only upon the choice of the ds^2 (or, what comes to the same when the complex-analytic structure is given, upon the choice of the associated form

$$\omega=\sum\frac{\partial^2\psi}{\partial z_i\partial\bar z_j}dz_i\wedge d\bar z_j,$$

where $\wedge$ denotes Grassmann multiplication) and upon the assumption that it is of Kähler type. Next, he observes in effect that the cohomology classes determined by the various types (in his classification) of effective or ineffective forms depend only upon the cohomology class determined by ω; this, for the particular metric used in his book, is the dual of the homology class of the hyperplane sections of the variety. Finally, he shows that the numbers of forms of the various types are even invariant by all one-to-one birational transformations. He also raises the following question: given a ds^2 of Kähler type, is it possible (with the above notations) to define ψ so that it will be regular everywhere except on an analytic subvariety, homologous to ω? It is stated without proof that, under some simple restrictive assumptions, this is indeed the case.

A. Weil.

Kasner, Edward, and DeCicco, John. Rational harmonic curves. Bull. Amer. Math. Soc. **53**, 824–831 (1947).

A polynomial harmonic curve is by definition a curve in the complex z-plane defined by setting the real part of a polynomial in z equal to zero: $\Re[p(z)]=0$. Polynomial harmonic curves have been studied and characterized. The authors now investigate rational harmonic curves, defined by $\Re[r(z)]=0$, where $r(z)$ denotes an arbitrary rational function of z. They obtain various geometric properties of these curves, generalizing previously known results concerning polynomial harmonic curves. Thus corresponding to the known result that the n asymptotes of a polynomial harmonic curve of degree n are all real, concurrent and symmetrically disposed about their common point, the authors establish the following result. A rational harmonic curve of degree r in $z=x+iy$ is given by an equation $P(x, y)=0$, where $P(x, y)$ is a polynomial of degree $n=2r-k$, with $0\leqq k\leqq r$; there are k asymptotes which are real, concurrent, and symmetrically disposed about their common point; the remaining $2(r-k)$ asymptotes are minimal.

E. F. Beckenbach (Los Angeles, Calif.).

Wakerling, R. K. The chordal hypersurfaces of a rational curve. Duke Math. J. **14**, 795–802 (1947).

L'auteur représente en premier lieu les ∞^2 droites, qui rencontrent en deux points une courbe rationnelle normale C^4 d'un espace S_4, de la manière suivante. Les ∞^1 hyperplans osculateurs de C^4 découpent sur un plan osculateur quelconque de C^4 les ∞^1 tangentes d'une conique γ; à une corde de C^4, qui rencontre la courbe C^4 en deux points A, B, on peut alors faire correspondre le point d'intersection des deux tangentes de γ qui appartiennent aux hyperplans osculateurs de C^4 en A, B. Cette représentation est biunivoque, en général. D'une manière analogue on peut représenter les ∞^{k+1} espaces S_k qui rencontrent en $k+1$ points une courbe rationnelle normale C^r d'un espace S_r sur

les points d'un espace S_{k+1} osculateur de C^r. L'auteur en deduit la représentation sur le plan (ou sur S_{k+1}) des sections hyperplanes de la variété formée par les cordes de C^4 (ou par les S_{k+1} qui contiennent $k+1$ points de C^r).

E. G. Togliatti (Gênes).

Lilley, S. On the construction of algebraic curve branches of given composition. J. London Math. Soc. **22**, 67–74 (1947).

The author gives an explicit method, based on a succession of quadratic transformations, for constructing algebraic curve branches (in space of any number of dimensions) with a singularity of given composition at the origin. The method, while not differing fundamentally from that used by the reviewer in a similar connection [same J. **21**, 233–240 (1947); these Rev. **8**, 598], is more explicit and the construction takes a somewhat neater form. *J. A. Todd.*

Franchetta, A. Sulla curva doppia della proiezione di una superficie generale dell'S_4, da un punto generico su un S_3. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **2**, 276–279 (1947).

In 1942 the author sought to show that a surface F of the space S_4, with only a finite number of improper double points, which is the projection F^1 of an S_3 having a reducible double curve, is a projection of the surface of Veronese. The proof given then was not entirely complete. In the present paper this proof is completed. *T. R. Hollcroft.*

***van der Waerden, B. L. The foundation of algebraic geometry. A very incomplete historical survey.** Studies and Essays Presented to R. Courant on his 60th Birthday, January 8, 1948, pp. 437–449. Interscience Publishers, Inc., New York, 1948. $5.50.

van der Waerden, Bartel L. Birational invariants of algebraic manifolds. Acta Salmanticensia. Ciencias: Sec. Mat., no. **2**, 56 pp. (1947).

The contents of this paper are on the main expository. In chapter I the notion of a simple point is discussed. The ground field is assumed to be of characteristic zero. In chapter II the differentials of the first kind are introduced and the corresponding definition of the irregularity and the geometric genus of an algebraic surface is developed. In chapter III valuation theory is used in order to define invariantively base conditions for linear systems of curves. [The author apparently has overlooked the close connection that exists between the material of chapter III and the reviewer's paper, Amer. J. Math. **60**, 151–204 (1938). One of the primary objects of that paper was precisely to "replace the intricate analysis of infinitely near points by valuation theory" and to provide through the concept of a "complete ideal" (= intersection of valuation ideals) a birationally invariant framework for linear systems defined by arbitrary base conditions.] *O. Zariski.*

d'Orgeval, Bernard. Les plans multiples représentatifs de certaines familles de surfaces algébriques. Bull. Soc. Math. France **74**, 87–101 (1946).

Casadio, G. Sulle trasformazioni puntuali che hanno direzioni inflessionali passanti per punti fissi. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **2**, 555–558 (1947).

The author investigates plane Cremona transformations for which the three (distinct or coincident) characteristic lines through a generic point P (i.e., lines whose homologues are inflexional at the corresponding point $\bar{P}$) pass always through fixed points. The only cases are the quadratic transformations and a particular type of de Jonquières transformation of arbitrary order. *J. G. Semple.*

Stubban, John Olav. Quelques recherches sur les transformations birationnelles dans la géométrie de direction. Arch. Math. Naturvid. **49**, no. 1, 1–18 (1947).

This paper is a generalization (to transformations of arbitrary order) of the author's previous work on quadratic birational transformations of the directed lines of a plane [same Arch. **48**, no. 3, 37–56 (1945); these Rev. **9**, 58]. If the directive lines of two planes π, π' are mapped on the points of two quadric cones (Bricard cones) K, K' of space, the transformations discussed are those corresponding to such birational mappings of K on K' as are subordinate to space Cremona transformations and carry the generators of K into those of K'. The order of any such transformation of π into π' is the number κ such that any direction curve of class n of π which touches no fundamental line is transformed into a curve of class κn in π'. By projecting K, K' from corresponding points, the author reduces the theory to that of plane de Jonquières transformations whose vertices represent the vertices of K, K' and the lines at infinity in π, π'. The paper contains numerous interpretations, in terms of direction geometry, of properties of de Jonquières transformations, including many special properties of the fundamental lines, properties of double lines (i.e., self-corresponding lines in the case when π, π' coincide), and a discussion of involutory transformations. *J. G. Semple.*

Differential Geometry

Kasner, Edward, and DeCicco, John. Harmonic transformation theory of isothermal families. Bull. Amer. Math. Soc. **53**, 832–840 (1947).

Isothermal properties of nonsingular harmonic transformations and of conformal transformations are obtained. It is shown that $T: u=\varphi(x, y)$, $v=\psi(x, y)$ is either a nonsingular harmonic transformation, or the product of a conformal map by a circle-to-line transformation, if and only if more than four pencils of parallel straight lines in the (u, v)-plane correspond to isothermal families in the (x, y)-plane; that the only isothermal families of curves in the (u, v)-plane which correspond to isothermal families in the (x, y)-plane under all nonsingular harmonic transformations are the pencils of straight lines; and that the only transformations T whereby every pencil of parallel straight lines and also every set of concentric circles in the (u, v)-plane corresponds to an isothermal family in the (x, y)-plane are the conformal transformations. *E. F. Beckenbach.*

Löbell, Frank. Flächenabbildungen mit gemeinsamem Invariantensystem. Math. Ann. **120**, 21–35 (1947).

Soient ${}^1\mathbf{r}(u, v)$, ${}^2\mathbf{r}(u, v)$ les radiusvecteurs réels de deux points correspondants dans la représentation ${}^1\mathbf{r} \rightleftarrows {}^2\mathbf{r}$ de deux surfaces. Les vecteurs $\mathbf{j}_1(u, v)={}^1\mathbf{r}_u \times {}^1\mathbf{r}_v$, $\mathbf{j}_2(u, v)={}^2\mathbf{r}_u \times {}^2\mathbf{r}_v$, $\mathbf{j}(u, v)={}^1\mathbf{r}_u \times {}^2\mathbf{r}_v - {}^1\mathbf{r}_v \times {}^2\mathbf{r}_u$ et le scalaire $j(u, v)={}^1\mathbf{r}_u{}^2\mathbf{r}_v - {}^1\mathbf{r}_v{}^2\mathbf{r}_u$ (où $\mathbf{r}_u=\partial\mathbf{r}/\partial u$, $\mathbf{r}_v=\partial\mathbf{r}/\partial v$) [dits les invariants (1)] forment le système complet des invariants différentiels du poids 1 des surfaces pendant les transformations des paramètres u, v. L'auteur s'occupe de la question suivante. Soit donné

le système complet des invariants (1) des surfaces $^1\mathbf{r}$, $^2\mathbf{r}$. Existe-t-il sauf $^1\mathbf{r}$, $^2\mathbf{r}$ une autre paire $^1\bar{\mathbf{r}}$, $^2\bar{\mathbf{r}}$ des surfaces avec le même système des invariants (1)? L'auteur étudie la correspondance $^1\mathbf{r}\rightleftarrows{}^1\bar{\mathbf{r}}$, $^2\mathbf{r}\rightleftarrows{}^2\bar{\mathbf{r}}$ et montre qu'ils existent aux points correspondants des surfaces $^1\mathbf{r}$, $^1\bar{\mathbf{r}}$ (respectivement $^2\mathbf{r}$, $^2\bar{\mathbf{r}}$) des vecteurs $^1\mathbf{v}={}^1\mathbf{r}_udu+{}^1\mathbf{r}_vdv$, $^1\bar{\mathbf{v}}={}^1\bar{\mathbf{r}}_udu+{}^1\bar{\mathbf{r}}_vdv$ (respectivement $^2\mathbf{v}$, $^2\bar{\mathbf{v}}$) qui sont parallèles. Trois cas peuvent se présenter: il existe (1) deux vecteurs diverses $^1\mathbf{v}_1$, $^1\mathbf{v}_2$ ou (2) deux vecteurs confondus $^1\mathbf{v}_1\equiv{}^1\mathbf{v}_2$ ou (3) dans un cas spécial on obtient le faisceau des vecteurs $^1\mathbf{v}_1+\lambda^1\mathbf{v}_2$.

A chacun de ces cas l'auteur trouve des conditions nécessaires et suffisantes pour que les surfaces $^1\bar{\mathbf{r}}$, $^2\bar{\mathbf{r}}$ existent. En partant de ces relations, les surfaces sont déterminées par l'intégration. On montre qu'aux certaines conditions dans le premier cas deux cas peuvent se présenter. (a) Il existe sauf $^1\mathbf{r}$, $^2\mathbf{r}$ seulement une paire $^1\bar{\mathbf{r}}$, $^2\bar{\mathbf{r}}$ des surfaces essentiellement différentes de la paire $^1\mathbf{r}$, $^2\mathbf{r}$ (c'est-à-dire, on ne peut pas obtenir cette paire par le mouvement euclidien en partant des surfaces $^1\mathbf{r}$, $^2\mathbf{r}$). (b) Il existe sauf $^1\mathbf{r}$, $^2\mathbf{r}$ un système à un paramètre des paires $^1\bar{\mathbf{r}}$, $^2\bar{\mathbf{r}}$ des surfaces. Dans le second cas ils existent sauf $^1\mathbf{r}$, $^2\mathbf{r}$ encore les paires $^1\bar{\mathbf{r}}$, $^2\bar{\mathbf{r}}$ avec le système des invariants différentiels (1). Lorsque les surfaces $^1\mathbf{r}$, $^2\mathbf{r}$ sont réglées et telles qu'à chaque droite de $^1\mathbf{r}$ correspond une droite de $^2\mathbf{r}$ dans une similitude qui varie en changeant la droite de $^1\mathbf{r}$, il existe une infinité de paires des surfaces $^1\bar{\mathbf{r}}$, $^2\bar{\mathbf{r}}$. Dans le troisième cas spécial on n'a pas des solutions essentiellement différentes de surfaces $^1\mathbf{r}$, $^2\mathbf{r}$. *F. Vyčichlo* (Prague).

Maneng, Louis. Sur les parties réelle et imaginaire des formes minima d'une surface. C. R. Acad. Sci. Paris **225**, 1115–1116 (1947).

Lalan [same C. R. **223**, 707–709 (1946); these Rev. **8**, 228] has introduced two conjugate complex differential forms ω_1 and ω_2 to be used in the study of the differential geometry of a surface. This note investigates the geometric interpretations of two real differential forms θ_1 and θ_2 which are to within a factor the real and imaginary parts of ω_1.

C. B. Allendoerfer (Haverford, Pa.).

Rozenfel'd, B. A. The metric and affine connection in spaces of planes, spheres or quadrics. Doklady Akad. Nauk SSSR (N.S.) **57**, 543–546 (1947). (Russian)

Many geometrical figures, among them points, planes, spheres, quadrics in various spaces, may be viewed from a common point of view as "figures of symmetry." The following are the spaces considered: a projective n-space P_n, Euclidean n-space R_n, an l-pseudo-Euclidean ^lR_n (with an indefinite metric of index l), an elliptic n-space S_n, an l-pseudo-elliptic space ^lS_n, a conformal n-space C_n, a complex unitary-elliptic space K_n and a double unitary elliptic space B_n (B_n differs from K_n in that complex numbers are replaced by Clifford numbers ($\alpha=a+\omega b$, $\omega^2=\pm1$). The fundamental groups of these spaces are denoted by corresponding German letters $\mathfrak{P}_n$ (projective collineations), $\mathfrak{R}_n$ (Euclidean motions), etc. In these spaces the figures considered are m-pairs in P_n (a configuration of an m-plane and an $(n-m-1)$-plane; m-planes in R_n, ^lR_n, S_n, ^lS_n, K_n and B_n; m-spheres in C_n and hyperquadrics in P_n. The complete manifolds of these figures are denoted by P_n^m, R_n^m, etc. and the last one by Q_n^{n-1}. Symmetries with respect to m-planes in R_n, ^lR_n, S_n, ^lS_n are classical: they are reflections. The general principle is given by theorem 1: the numerical and geometric invariants of two figures of symmetry are invariants of that transformation of the fundamental group which is the product of the symmetries with respect to these figures; they are the invariant figures of this transformation and the characteristic roots of the matrix of this transformation. The main theorem of the paper proves that in a space of figures of symmetry it is possible to introduce an affine connection (without torsion) which is invariant under the fundamental group. If the group is semisimple a Riemannian or a pseudo-Riemannian metric may be introduced (Riemannian if the group is compact; all these groups, except those of R_n and ^lR_n, are semisimple; those of C_n and K_n are compact). The final result of the paper is that the geodesics (or paths) in these spaces correspond to one-parameter subgroups of the fundamental group.

M. S. Knebelman (Pullman, Wash.).

Ruse, H. S. Multivectors and catalytic tensors. Philos. Mag. (7) **38**, 408–421 (1947).

Two of the results for self-dual six-vectors of null invariant in Galilean space-time which E. T. Whittaker [Proc. Roy. Soc. London. Ser. A. **158**, 38–46 (1937)] demonstrated with the help of spinor theory are here demonstrated by direct geometric methods, without the use of spinor theory, and for a Riemannian V_4 of any signature and curvature. Light is thrown on the nature of catalytic tensors, and we have an illustration of the value of geometric methods in discovering tensor formulae which involve covariant derivatives. The extension of the results to higher dimensions is indicated, using a tensor notation like that of J. W. Givens [Ann. of Math. (2) **38**, 355–385 (1937)]. *A. Schwartz.*

De Donder, Th., et Van Isacker, J. Contribution à la théorie des transformations infinitésimales des spineurs. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **33**, 280–287 (1947).

One can solve the infinitesimal transformation equations

$$dx^i/d\tau=X^i(\tau;x^1,\cdots,x^n),\tag{1}$$

where the X^i are uniform and continuous with respect to all their variables, to get $x^i=f^i(\tau;x_0^1,x_0^2,\cdots,x_0^n)$, with $x^i=x_0^i$ for $\tau=\tau_0$, and one can then define the finite transformation for a contravariant vector:

$$A^i=\frac{\partial f^i(\tau;\overset{1}{x_0},\cdots,\overset{n}{x_0})}{\partial \overset{j}{x_0}}A_0^j.$$

One can generalize this definition to read

$$y^\alpha=\frac{\partial f^i(\tau;x_0)}{\partial \overset{j}{x_0}}e_{i\beta_\alpha}^{j\alpha}\overset{\beta}{y_0},$$

where the e's do not depend on τ, if the transformations (1) are restricted; for instance, if they generate a finite and continuous group. Thus, if the Euclidean metric is preserved, one is led to tensors and the classical spinors. If the metric defined by

$$g_{ij}=\frac{\delta_{ij}}{U}-\frac{x^ix^j}{U^2},\quad U=1+\sum_k(x^k)^2,$$

is preserved one is led to a generalized spinor.

A. Schwartz (State College, Pa.).

NUMERICAL AND GRAPHICAL METHODS

***Cazzola, E. Tavole Grafiche dei Logaritmi a 6 Decimali con Interpolazione Ottica sulla 6ª Cifra. Tavole Aritmetiche e Numeriche.** Ulrico Hoepli, Milano, 1947. xix+90 pp. 1000 lire.

The main part of the book [pp. 4–63] consists of a series of vertical strips resembling the scales of an elongated slide rule, from which one can read five-figure logarithms or cologarithms of five-figure numbers and inversely; it is possible to interpolate by eye for the sixth figures. [A table of similar scope by A. Lacroix and C. L. Ragot ["A Graphic Table Combining Logarithms and Anti-Logarithms," Macmillan, New York, 1925] has the scales arranged horizontally and of a somewhat larger size, but does not provide for direct reading of cologarithms.] The remainder of the book consists of brief miscellaneous tables: conversion from $\log_{10}$ to and from $\log_e$; squares, cubes, square roots, cube roots, $e^{\pm n/100}$, πn, etc., for n from 1 to 1000; special constants involving π and e; etc. *R. P. Boas, Jr.* (Providence, R. I.).

Lubkin, Samuel. Decimal point location in computing machines. Math. Tables and Other Aids to Computation **3**, 44–50 (1948).

Brainerd, J. G., and Sharpless, T. K. The ENIAC. Elec. Engrg. **67**, 163–172 (1948).

Cf. Goldstine and Goldstine, Math. Tables and Other Aids to Computation **2**, 97–110 (1946); these Rev. **8**, 354.

Burks, Arthur W. Electronic computing circuits of the ENIAC. Proc. I. R. E. **35**, 756–767 (1947).

The ENIAC, the first electronic general-purpose digital machine to be built, consists of some 18,000 tubes with proportionately many other parts. It is clear that the various circuits must be extremely reliable if such a machine is to operate successfully. In view of this two principles were followed: "carefully selected and rigidly tested standard components which are operated well below their normal ratings" were used, and all the circuits were designed to have wide tolerances. This paper illustrates the carrying out of these principles in a number of typical circuits of the ENIAC. *R. W. Hamming* (Murray Hill, N. J.).

van der Pol, Balth. An electro-mechanical investigation of the Riemann zeta function in the critical strip. Bull. Amer. Math. Soc. **53**, 976–981 (1 plate) (1947).

Starting with the infinite integral for $\zeta(s)$, where $\Re(s)>1$ [Whittaker and Watson, A Course of Modern Analysis, 4th ed., Cambridge University Press, 1927, p. 266] the author obtains several relations for $\zeta(s)$ including the integral

$$(1) \qquad \zeta(s)/s=\int_0^\infty ([u]-u)u^{-s-1}du, \qquad 0<\Re(s)<1.$$

[Compare van der Pol, Philos. Mag. (7) **26**, 921–940 (1938).] Setting $u=e^x$ and $s=a+it$ in (1) gives a Fourier transform representation.

This result was applied to investigate the zeros in the critical strip by taking $a=\frac{1}{2}$ and approximating the moduli of more than 600 harmonics of the Fourier series for the portion of the function $e^{x/2}-e^{-x/2}[e^x]$ between $x=-9$ and $x=+9$. This was accomplished by means of an electro-mechanical harmonic analyzer of heterodyne type in which the function to be analyzed was introduced by optical means [for such instruments see H. H. Hall, J. Acoust. Soc. Amer. **8**, 257–262 (1937)]. The investigation so far has been confined to the line $\Re(s)=\frac{1}{2}$ so as to test the experimental technique. The results are in good agreement with what is known of the zeros on this line as far as $t=210$. *R. Church* (Annapolis, Md.).

Bleick, W. E. Calculating machine solution of quadratic and cubic equations by the odd number method. Math. Tables and Other Aids to Computation **2**, 321–324 (1947).

Generalizes the well-known method of extracting square roots by the odd number method to cube roots and to quadratic and cubic equations. *E. Bodewig* (The Hague).

Bodewig, E. Bericht über die verschiedenen Methoden zur Lösung eines Systems linearer Gleichungen mit reellen Koeffizienten. I. Nederl. Akad. Wetensch., Proc. **50**, 930–941 = Indagationes Math. **9**, 441–452 (1947).

After a general discussion of the difficulties involved in the solution of simultaneous equations in many unknowns the author proceeds to an analysis of the number of operations required by various methods of solution. He concludes that for n unknowns: (1) Gauss's method requires $\frac{1}{3}n(n^2-1)+n^2$ multiplications, $\frac{1}{6}n(n-1)(2n+5)$ additions; (2) Cholesky's method (applicable only to symmetric systems) requires n square roots, $\frac{1}{6}n(n^2+9n+2)$ multiplications, $\frac{1}{6}n(n-1)(n+7)$ additions; (3) by Schur's method the calculation of the inverse matrix requires n^3 multiplications, n^3-2n^2+2n additions; the solution of the system of equations requires in addition n^2 multiplications, n^2-n additions. *W. E. Milne* (Corvallis, Ore.).

Frazer, R. A. Note on the Morris escalator process for the solution of linear simultaneous equations. Philos. Mag. (7) **38**, 287–289 (1947).

The author explains Morris's process by using matrix symbolism (in the case of a symmetrical matrix) and recommends the process. However, the reviewer has calculated the number of multiplications necessary for the solution of a system of n equations and found that Morris's process requires about $n/4$ times as many multiplications as the usual Gauss method. For an unsymmetrical matrix the author recommends Morris's device of solving two systems of equations or the "normalisation" of the system and then the solution by Morris's process. However, both devices are again inconvenient. For example, the normalisation alone, which is only the first part of the solution, requires three times as many multiplications as the whole solution by Gauss's method. *E. Bodewig* (The Hague).

Platone, Giulio. Ancora sul metodo delle corde per la risoluzione numerica dei sistemi di equazioni. Rend. Circ. Mat. Palermo **62**, 363–368 (1941).

The author proves that the method of chords, generalised to systems of equations, is convergent. Let in $y=f(x)$, $y=\varphi(x)$ the functions f, φ be continuous in the interval (a_1, b_1) and let, for example, $f(a_1)<\varphi(a_1)$, $f(b_1)>\varphi(b_1)$. Then on each of the curves the chord is drawn between its end points. The abscissa x_1 of the intersection of the two chords is chosen as the end point to the left or to the right of a new interval (a_2, b_2), according as $f(x_1)$ is less or greater

than $\varphi(x_1)$. In the new interval the procedure is repeated, and the author proves that the intervals (a_n, b_n) will decrease to zero and yield an intersection point X of the curves f, φ. The error of the nth approximation x_n is

$$X-x_n=-\frac{(b_n-a_n)(b_n-X)(X-a_n)F''(x)}{F(b_n)-F(a_n)},$$

where $F(x)=f(x)-\varphi(x)$ and $a_n<x<b_n$, so that, if $M=\max|F''(x)|$, $|X-x_n|<\frac{1}{8}(b_n-a_n)^3M/|F(b_n)-F(a_n)|$. The case when x, y are polar coordinates is reduced to the case of rectangular coordinates by a central projection.

E. Bodewig (The Hague).

Chernoff, H. A note on the inversion of power series. Math. Tables and Other Aids to Computation **2**, 331–335 (1947).

Treats the multiplication of power series and their inversion by means of a movable strip of paper on which the coefficients are written. *E. Bodewig* (The Hague).

Salzer, Herbert E. Tables for facilitating the use of Chebyshev's quadrature formula. J. Math. Phys. Mass. Inst. Tech. **26**, 191–194 (1947).

Gives the roots z_i of the Chebyshev polynomials of degrees 1 to 7 and 9 to 10 decimal places. Furthermore, the coefficients of the linear function of $f(z_i)$ which is the $(n-1)$th divided difference of the $f(z_i)$, $i=1, \cdots, n$. This is used for a check of the integrand of Chebyshev's quadrature formula $\int_{-1}^{1}f(z)dz=2n^{-1}\sum_1^n f(z_i)+R_n$. *E. Bodewig.*

Tolstov, Yu. G. An electrical device for the solution of homogeneous and inhomogeneous ordinary linear differential equations of higher order with constant coefficients, giving the solution in the form of a Taylor series. Bull. Acad. Sci. URSS. Cl. Sci. Tech. [Izvestia Akad. Nauk SSSR] **1947**, 319–322 (1947). (Russian)

If the equation $L(D)y=f(t)$, where $f(t)=\sum f^i(\tau)(t-\tau)^i/i!$, has a power series solution $y(t)=\sum Y_i(t-\tau)^i/i!$, then the coefficients Y_i satisfy $L(\Delta)Y_i=f^i(\tau)$, where $\Delta^jY_i=Y_{i+j}$. If Y_0 to Y_{n-1} are given, where $L(D)$ is of order n, then Y_{n+i}, $i=0, 1, 2, \cdots$ may be found from (*) $Y_{n+1}=a_n^{-1}[f^i(\tau)-M(\Delta)Y_i]$, where $L(\Delta)=a_n\Delta^n+M(\Delta)$. An electrical network containing two sets of n potentiometers, cyclically commutated against each other, is described, which can solve (*) with two hand-settings per value of i, when multiplication is replaced by approximate multiplication by two cascaded potentiometers. [The circuit would also apply to de Moivre's method of solving algebraic equations.]

J. W. Tukey (Princeton, N. J.).

Poritsky, H., and Blewett, M. H. A method of solution of field problems by means of overlapping regions. Quart. Appl. Math. **3**, 336–347 (1946).

At the outset the authors outline a practical procedure for the solution of Laplace's equation for a plane region with assigned boundary values by means of successive approximations based on Schwarz's "alternating procedure." The method is then applied with suitable modifications to the solution of the two-dimensional wave equation for the propagation of waves around a right angled corner. The region in question is L-shaped with both legs of the L extending to infinity and both of the same width. In the solution advantage is taken of the principle of reflection. Appropriate assumptions are made regarding the character of the waves at infinity. *W. E. Milne* (Corvallis, Ore.).

Jaeger, J. C., and Clarke, Martha. Numerical results for some problems on conduction of heat in slabs with various surface conditions. Philos. Mag. (7) **38**, 504–515 (1947).

Eight basic functions of two dimensionless variables are defined by means of infinite series and displayed graphically in such a manner that their values can be read off with considerable accuracy. Ten different types of boundary value problems for temperatures in slabs with one of the faces either subject to the linear law of heat transfer from the surroundings or placed in contact with a perfect conductor are then described. The unknown variable temperature and flux at the faces in those ten types of problems are written as simple linear combinations of the eight functions. The authors also point out how the numerical values of temperature and flux at the faces, as well as the total heat content, can be read off for any time in certain other types of problems. They show how this numerical method supplements the methods of other authors. *R. V. Churchill.*

Hartley, H. O. The application of some commercial calculating machines to certain statistical calculations. Suppl. J. Roy. Statist. Soc. **8**, 154–173; discussion, 173–183 (4 plates) (1946).

Examples of highly-developed technique in using tape adding machines, desk calculators and punched card equipment. Application to the calculation of sums of squares and cross-products, moving averages, serial correlations, simultaneous equations. Numerical integration with the National accounting machine. *J. W. Tukey* (Princeton, N. J.).

Zwinggi, Ernst. Über Darstellungsformen der Prämien und Reserven der Todesfallversicherung. Mitt. Verein. Schweiz. Versich.-Math. **47**, 409–413 (1947).

The author derives an integral equation for the (continuously payable) premium and for the policy value of a whole life insurance. *E. Lukacs* (Cincinnati, Ohio).

MECHANICS

Hydrodynamics, Aerodynamics, Acoustics

Gurevič, M. I. Remarks on F. Vasilesco's papers concerning axisymmetrical flows with free boundaries. Doklady Akad. Nauk SSSR (N.S.) **57**, 763–764 (1947). (Russian)

F. Vasilesco [C. R. Acad. Sci. Paris **196**, 896–898, 1284–1286 (1933)] gave without proof a method of reducing an axisymmetric flow with free streamlines to a two-dimensional flow with streamlines identical with those in a meridian plane of the axisymmetric flow but with the free streamline boundary condition replaced by another condition. In the present note the author shows in two simple exampies that Vasilesco's method must be incorrect. *J. V. Wehausen.*

Küchemann, D. Tafeln für die Stromfunktion und die Geschwindigkeitskomponenten von Quellring und Wirbelring. Jahrbuch **1940** der Deutschen Luftfahrtforschung, I547–I564 (1940).

To provide data for engineering calculations involving hollow figures of revolution, tables are provided giving the

stream function and the radial and axial velocity components due to source-rings and vortex-rings in incompressible ideal-fluid flow. The formulas for these quantities involve elliptic integrals. The ring is located at $x=0$ and has the radius r'; the tables cover the range $0<x/r'<10$ and $0<r/r'<10$. The potential for dipole-rings, having dipole axes parallel and perpendicular to the plane of the ring, can also be read from these tables. Drawings are provided showing graphically the contours of the functions tabulated. Similar problems are encountered in the field of electromagnetic theory. No mention is made here of any analogous tabulations in that literature. *W. R. Sears* (Ithaca, N. Y.).

Gilbarg, D. On the flow patterns common to certain classes of plane fluid motions. J. Math. Phys. Mass. Inst. Tech. **26**, 137–142 (1947).

It is proved that (i) the stream functions of any two steady plane liquid flows that possess the same streamlines must be connected by a linear relation, except for the special case of those flows having constant velocity along the streamlines; (ii) the only flow patterns common to the potential flows of gases and liquids are those corresponding to circular vortex and radial flows. *A. J. McConnell.*

Zenkin, A. The flow around a sphere in the presence of a vortex ring. Doklady Akad. Nauk SSSR (N.S.) **58**, 373–375 (1947). (Russian)

The author finds the stream function for the flow of an ideal fluid about a sphere of radius a arising from a uniform flow and a vortex ring of intensity Γ and radius R with center at distance Z_1 from the center of the sphere, the plane of the ring being perpendicular to the uniform flow. *J. V. Wehausen* (Falls Church, Va.).

Nekrasov, A. I. Survey of the author's work on aerohydromechanics. Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1947**, no. 10, 1265–1270 (1947). (Russian)

Shapiro, Ascher H., and Hawthorne, W. R. The mechanics and thermodynamics of steady one-dimensional gas flow. J. Appl. Mech. **14**, A-317–A-336 (1947).

Edelman, G. M., and Shapiro, Ascher H. Tables for numerical solution of problems in the mechanics and thermodynamics of steady one-dimensional gas flow without discontinuities. J. Appl. Mech. **14**, A-344–A-351 (1947).

Turner, L. Richard, Addie, Albert N., and Zimmerman, Richard H. Charts for the analysis of one-dimensional steady compressible flow. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1419**, 38 pp. (23 plates) (1948).

Schäfer, Manfred. Equations for adiabatic but rotational steady gas flows without friction. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **1187**, 23 pp. (1947).

[Translation of Lehrstuhl für Technische Mechanik an der Technischen Hochschule Dresden, Archiv no. **44/1.**] The author derives the differential equation satisfied by the stream function of a steady, isoenergetic compressible flow. These assumptions allow for the presence of vorticity, and in consequence for a change of entropy from streamline to streamline. The results are given explicitly for two-dimensional and rotationally symmetric flows. The recent literature on this and related topics is exhaustively discussed in the course of the paper and in an appendix.
D. P. Ling (Murray Hill, N. J.).

Ward, G. N. A note on compressible flow in a tube of slightly varying cross-section. Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda no. **2183** (**9205**), 7 pp. (1945).

Non-uniform compressible flow in a tube is investigated using the approximate linearized equation for the velocity potential. It is found that for supersonic flow, stationary oscillatory disturbances result from initially non-uniform conditions and these persist down the tube. For subsonic flow in a straight tube no oscillatory disturbances occur and any initial non-uniformity is damped out downstream in general. A method of calculating the supersonic flow in a tube of slightly varying cross-section is indicated.
Author's summary.

Göthert, B. Ebene und räumliche Strömung bei hohen Unterschallgeschwindigkeiten. Jahrbuch **1941** der Deutschen Luftfahrtforschung, I156–I157 (1941).

The author notes that in the usual linear-perturbation theory of compressible flow (the Prandtl-Glauert approximation) the affine distortion of the profile is inconsistent with the distortion of the velocity field. [He does not mention that in the plane case this inconsistency is permissible, while in an axisymmetric case it is not.] He proposes to resolve this difficulty by using the following transformation of velocity potential from incompressible $(\ )_1$ to compressible $(\ )_2$ flow: $\phi_1=Ux+f(x,y,z)$, $\phi_2=\beta^2Ux+f(x,\beta y,\beta z)$, where U denotes the stream speed (in the x direction) and β^2 denotes $1-M^2$, M being the stream Mach number. Proceeding to the consideration of a finite-span wing of aspect ratio Λ, according to the Prandtl lifting-line theory, the author determines that the slopes of the lift curves for compressible and incompressible flow are in the ratio $(1.8+\Lambda)/(1.8+\beta\Lambda)$, where the value $1.8\pi\approx 5.65$ has been taken for the profile constant. *W. R. Sears.*

Lamla, Ernst. Über die ebene Potentialströmung um ein Profil, das sich konform auf einen Kreis abbilden lässt, im unterkritischen Gebiet. Jahrbuch **1940** der Deutschen Luftfahrtforschung, I26–I35 (1940).

It is shown how the Janzen-Rayleigh approximation in the case of steady flow past a cylinder can be reduced, by conformal transformation, to a similar problem involving a circular boundary. In particular, the author considers transformations of the form

$$\frac{z+k\mu R}{z-k\lambda R}=\left(\frac{\zeta+\mu R}{\zeta-\lambda R}\right)^k$$

which can be used to map several useful profiles on a circle of radius R in the ζ-plane. The constants μ and λ are complex, in general, while k is real. For such transformations the calculation of the first correction term (proportional to M^2) is carried through in detail. In the special case of an elliptic cylinder ($k=2$, $\mu=\lambda$) the result agrees with Kaplan's [Tech. Rep. Nat. Adv. Comm. Aeronaut., no. **624** (1938)]. The symmetrical Joukowski profile is considered next ($k=2$, $\mu=1$, $\lambda<1$), in symmetrical flow. Again, only the first correction term is calculated. The same is then done for the symmetrical Kármán-Trefftz profile, for which $k<2$, $\mu=1$, $\lambda<1$. The calculations for the latter case are quite lengthy, and numerical integration is resorted to in some steps.

Finally, numerical calculations are carried out for ellipse, Joukowski, and Kármán-Trefftz profiles, all of thickness ratio 0.1 and at zero incidence. The stream Mach numbers investigated are between 0 and 0.666. Velocity distributions at the surface are tabulated. *W. R. Sears.*

Kaplan, Carl. On the use of residue theory for treating the subsonic flow of a compressible fluid. Tech. Rep. Nat. Adv. Comm. Aeronaut., no. **728**, 39–47 (1942).

In the familiar Janzen-Rayleigh theory the velocity potential of a steady isentropic subsonic flow is expressed as $\phi=\phi_0+\phi_1M^2+\phi_2M^4+\cdots$, where M denotes the stream Mach number. The successive functions $\phi_1, \phi_2, \cdots$ can be considered to be the velocity potentials for an incompressible fluid with given distributions of source strength. Following Poggi [Aerotecnica **14**, 532–550 (1934)] the author presupposes a conformal transformation of the region exterior to a given obstacle in the z-plane into that exterior to a circle in a Z-plane. Poggi's expression for the complex velocity then appears as the sum of certain surface integrals, carried over the entire region exterior to the circle, involving the appropriate source distribution. Each successive complex velocity, corresponding to $\phi_1, \phi_2, \cdots$, respectively, is given by such integrals involving the preceding complex velocities. The author employs results of Milne-Thomson [Theoretical Hydrodynamics, Macmillan, London, 1938, p. 238] to convert these area integrals into contour integrals, after which he proceeds to evaluate these by the method of residues. He works out in detail the expression for W_1, the complex velocity (in the Z-plane) corresponding to ϕ_1, which depends upon the incompressible flow pattern. The application of this procedure to the particular case of an elliptic cylinder is shown. *W. R. Sears.*

Kaplan, Carl. On a new method for calculating the potential flow past a body of revolution. Wartime Rep. Nat. Adv. Comm. Aeronaut., no. **L-558**, 45 pp. (2 plates) (1942).

Kaplan, Carl. On a new method for calculating the potential flow past a body of revolution. Tech. Rep. Nat. Adv. Comm. Aeronaut., no. **752**, 13 pp. (1943).

Consider a body of revolution moving in the direction of its axis through an incompressible ideal fluid otherwise at rest. Let x, ω be the usual cylindrical coordinates and define $z=x+i\omega$, $\bar{z}=x-i\omega$. The author shows that the method of conformal transformation can be used to map the region exterior to the given profile in a meridional plane onto the region exterior to a circle of radius R in a Z-plane. If $Z=Re^{i\zeta}$ and $\zeta=\xi+i\eta$, it is seen that $\eta=0$ defines the circle; thus ξ and η are considered as a particular set of orthogonal curvilinear coordinates in the original meridional plane, such that the curve $\eta=0$ coincides with the profile of the body. This makes the boundary condition at the surface particularly simple. Moreover, the differential equation for the velocity potential ϕ is invariant in form in such transformations. The author sets up an iteration procedure by which ϕ is calculated in the form $\phi=\phi_0+\phi_1+\phi_2+\cdots$, assuming the constants of the conformal transformation are known. The successive contributions $\phi_0, \phi_1, \cdots$ are found in terms of Legendre functions P_n and Q_n; ϕ_0, ϕ_1, and ϕ_2 are worked out in detail. Each approximation requires a number of quadratures. The author proposes the method of Theodorsen and Garrick [same Rep., no. **452** (1933)] for the conformal transformation of an arbitrary profile into a circle. The convergence of the iteration process will depend on the magnitudes of the terms in the expression $z=Z+c_1+a_1Z^{-1}+a_2Z^{-2}+\cdots$ which represents the transformation.

As a numerical example, the calculation of the pressure distribution is carried out in detail for a shape whose meridional section is a Joukowski profile with rounded tail. The process is carried through ϕ_2. Results are also presented for another body whose section is a "low-drag airfoil" profile. The author states that a similar theory can be developed for calculation of transverse flow past bodies of revolution. *W. R. Sears* (Ithaca, N. Y.).

Schmieden, C. Die Berechnung kompressibler Unterschallströmungen mit Hilfe der Stromfunktion. Jahrbuch **1941** der Deutschen Luftfahrtforschung, I8–I10 (1941).

The author believes there are advantages in using the stream function ψ rather than the velocity potential in carrying out the Janzen-Rayleigh approximation. Therefore he works out the formula for ψ_2, where $\psi=\psi_1+M^2\psi_2+\cdots$, M being the stream Mach number, and calculates thereby the case of an elliptic cylinder at angle of incidence. This problem was treated earlier by other authors who used the velocity potential. *W. R. Sears* (Ithaca, N. Y.).

Schmieden, C. Die kompressible Strömung um ein Rotationsellipsoid nach der Methode von Janzen-Rayleigh. Jahrbuch **1942** der Deutschen Luftfahrtforschung, I72–I79 (1942).

Here the method of expanding the stream function in powers of M^2 (M being the stream Mach number) is applied to the axisymmetric case of flow past an ellipsoid of revolution. Only the first correction term, proportional to M^2, is calculated. The usual ellipsoidal coordinates ξ, η are introduced, where $-1<\xi<1$ and $\eta^2>1$. The expression for the correction term is then expanded in descending powers of η, and terms beyond η^{-13} are neglected; this restricts the theory to ellipsoids that are sufficiently nearly spherical. In the limiting case of the sphere the result is found to agree with Lamla's [same Jahrbuch **1939**, I165ff.]. Computations of maximum velocity ratio were carried out for four thickness ratios between 0.8 and 1 (sphere). The author states that Lamla's results, using more terms, showed the M^2 term to be about $\frac{2}{3}$ of the entire correction for the sphere at critical Mach number (i.e., at the M where the maximum local velocity is sonic). He therefore proposes to increase all his computed corrections 50 percent in estimating critical Mach numbers of thick ellipsoids. This rough procedure provides a curve of critical M against thickness ratio, for ellipsoids of revolution. *W. R. Sears* (Ithaca, N. Y.).

Wittich, H. Bemerkungen zur Druckverteilungsrechnung nach Theodorsen-Garrick. Jahrbuch **1941** der Deutschen Luftfahrtforschung, I52–I57 (1941).

The use of the Theodorsen-Garrick technique for determining the pressure distribution about an airfoil requires the evaluation of the Poisson integral

$$2\pi f(\varphi)=-\int_0^{2\pi} y(\theta)\cot[(\theta-\varphi)/2]d\theta.$$

Here y is usually known only numerically. One may replace y by a trigonometric sum in an approximate manner and integrate numerically. The present paper contributes the observation that if f is evaluated at appropriate points φ_i, the computation can be simplified. In fact, one can

write f as the finite sum $f(\varphi_\lambda)=\sum_n \alpha_{\lambda n} S_n$, where the $\alpha_{\lambda n}$ are determined once and for all and the S_n are associated with $y(\theta)$. The $\alpha_{\lambda n}$ are tabulated in the paper for one of the appropriate φ_j spacings. *G. F. Carrier* (Providence, R. I.).

Galin, L. A. Notes on the theory of a wing of finite span in a supersonic flow. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 11, 383–386 (1947). (Russian. English summary)

The author considers a supersonic flow around an airfoil of finite span. If we assume that the motion takes place in the x-direction, then the potential function φ satisfies the simplified compressibility equation $\varphi_{zz}+\varphi_{yy}-(M^2-1)\varphi_{xx}=0$ and the boundary conditions $\partial\varphi/\partial z=0$ for $z=0$ in front of the wing (inside the Mach cone). For the value of z belonging to the intersection S of the wing (and lying inside the Mach cone) with the (x, y)-plane, we have $\partial\psi/\partial z=f(C, y)$, $f(x, y)=\partial F(x, y)/\partial x$, where $z=\pm F(x, y)$ is the equation of the wing. Using the results on representation of solutions of the wave equation, the author obtains an approximate solution in the form

$$\varphi(x, y, z)=\pi^{-1}\int\int_S f(\xi, \eta)\{(x/(M^2-1)^{\frac{1}{2}}-\xi)^2 -(y-\eta)^2-z^2\}^{-\frac{1}{2}}d\xi d\eta.$$

In a similar manner the problem of the vibrating wing is treated. In the latter case the potential is assumed in the form $\varphi(x, y, z)+\Re\Phi(x, y, z)e^{i\Omega x-i\omega t}$, where ϕ satisfies the equation $\phi_{yy}+\phi_{zz}-\phi_{xx}-\lambda^2\phi=0$, $\lambda=$constant. Using Hadamard's formulas for the potential of a simple layer for the above equation, the author obtains an approximate formula satisfying the derived boundary conditions.

S. Bergman (Cambridge, Mass.).

Galin, L. A. A wing rectangular in plane in a supersonic flow. Akad. Nauk SSSR. Prikl. Mat. Meh. 11, 465–474 (1947). (Russian. English summary)

[A more accurate translation of the title would be Wings of rectangular planform in supersonic flow.] Part one discusses the problem of finding the velocity potential of supersonic flow around a stationary rectangular wing of arbitrary twist and camber. The solution is constructed from Busemann's formula for the flat lifting wing. Explicit expressions are obtained for the case that the surface of the wing is expressed as a polynomial in the variables in the plane of the wing. Part two treats the more general nonstationary problem of a harmonically vibrating wing. The author shows how solutions for such problems may be obtained from solutions of stationary problems. This general theory is then applied to the rectangular wing.

P. A. Lagerstrom (Pasadena, Calif.).

Galin, L. A. Impact on a solid body lying on the surface of a compressible fluid. Akad. Nauk SSSR. Prikl. Mat. Meh. 11, 547–550 (1947). (Russian)

In this paper there is considered the effect of an impact on a solid body which is bounded from below by the plane of the surface of a compressible fluid. It is assumed that the initial position and velocity of the solid body are given, and that after the impact the body is translated parallel to itself. The exact solution of the problem of the simultaneous motion of the body and the elastic medium after the impact is found. The solution is given for a period of time which the author defines as "the initial period of impact," whose length is determined. The displacement function, which gives the motion of the solid body during the initial period of impact, is shown to satisfy a homogeneous second order linear differential equation with constant coefficients. If the characteristic roots of this differential equation are real the body will simply sink into the fluid, if they are imaginary the body will at first sink and then rebound during the initial period of impact. There are some typographical errors in the paper, which are, however, easily corrected by taking recourse to one of the author's earlier papers to which reference is made [see the preceding review].

H. P. Thielman (Ames, Iowa).

Faĭnzil'ber, A. M. On the solution of the equations of motion of a viscous gas by quadratures. Doklady Akad. Nauk SSSR (N.S.) 57, 439–442 (1947). (Russian)

By introduction of the variables

$$y=u/(2c_pT_0)^{\frac{1}{2}}, \quad z=\{2\mu_0(2c_pT_0)^{\frac{1}{2}}\}^{-\frac{1}{2}}p^{1-n(k-1)/k}\tau$$

and

$$P=(k/(k-1))p^{-(k-1)/(2km)},$$

the author reduces the equations for the boundary layer in a compressible fluid to the form

$$z^2\partial^2 z/\partial y^2-mP'(1-y^2)^n\partial z/\partial y-Py(1-y^2)^{n-1}\partial z/\partial x=0.$$

Here the subscript 0 refers to conditions at the wall, n is the experimental constant in the formula $\mu/\mu_0=(T/T_0)^n$, k the adiabatic index and $m=\frac{1}{2}(k-1)/[(2n+1)k-2n]$. Solutions of the form $z=A(x)B_n(y)$ are investigated. The resulting nonlinear ordinary differential equation for B_n has a simple first integral. The author applies his method both to two-dimensional flow and to bodies of revolution.

P. A. Lagerstrom (Pasadena, Calif.).

Lapuk, B. B. The motion of a real gas in a porous medium. Doklady Akad. Nauk SSSR (N.S.) 58, 377–380 (1947). (Russian)

Hill, R., and Pack, D. C. An investigation, by the method of characteristics, of the lateral expansion of the gases behind a detonating slab of explosive. Proc. Roy. Soc. London. Ser. A. 191, 524–541 (1947).

The method of characteristics is applied to the study of the expansion behind the detonation wave moving along an uncased charge of explosive. The problem is assumed to be two-dimensional, the section of the block of explosive being a rectangle. The flow behind the detonation wave is considered to be potential flow, and the usual hydrodynamical equations in two dimensions are used. Special attention must be given to the region at the face of the charge where the characteristic net collapses. Here the velocity potential and density are assumed to be analytic, and series expansions are employed to obtain boundary conditions at some distance from the charge. The pressure and velocity distributions in the gaseous products of the explosion are described, and the shape and position of the shock wave set up ahead of the detonation wave are calculated. The connection with the simpler case of a semi-infinite block of explosive is also indicated.

E. N. Nilson (East Hartford, Conn.).

Stanyukovič, K. P. The motion of the particles of the products of detonation of a linear charge. Doklady Akad. Nauk SSSR (N.S.) 58, 763–766 (1947). (Russian)

Heins, A. E., and Feshbach, H. The coupling of two acoustical ducts. J. Math. Phys. Mass. Inst. Tech. 26, 143–155 (1947).

The authors deal with acoustic plane waves propagating inside a wave guide of rectangular cross-section. The guide is assumed infinitely long in the x-direction. The vertical walls are perfectly rigid, while the horizontal are covered with acoustic material of nonzero "admittance." Further, the admittance for $x<0$ is different from that for $x>0$. An expression is given for the coefficient of reflection of an incoming plane wave reflected by the discontinuity at $x=0$. The physical problem is reduced to the solution of a Wiener-Hopf integral equation. This equation is solved by Fourier transforms. *C. J. Bouwkamp* (Eindhoven).

Elasticity, Plasticity

Friedrichs, K. O. On the boundary-value problems of the theory of elasticity and Korn's inequality. Ann. of Math. (2) 48, 441–471 (1947).

The paper deals with the boundary-value and eigenvalue problems of elasticity by means of direct variational methods. The tools used here (such as strong and weak derivatives and the mollifiers) have been introduced by the author in previous papers [e.g., Trans. Amer. Math. Soc. 55, 132–151 (1944); these Rev. 5, 188]. The key for the application of the variational method is Korn's inequality $D(u)<KE(u)$, where $D(u)$ is the Dirichlet integral for the displacement vector u and $E(u)$ is the strain energy. An alternative form of this inequality has been already given by Korn himself. The author's proof of Korn's inequality is much more concise than the original work by Korn. The boundary-value problems are treated as problems in Hilbert space which is completed by the introduction of ideal functions. The existence of a solution is then proved with comparative ease. However, an important point remains to be shown: that the solution is a "real" function satisfying the differential equation, and not merely an ideal function. This is done by the author without the use of the fundamental solution or Green's function, which were used by others in similar problems. The author strongly advocates the introduction of ideal functions, a point on which the reviewer agrees wholeheartedly inasmuch as this procedure (which is equivalent to the usual one) does not assume a knowledge of Lebesgue theory and thus makes the use of Hilbert space accessible to many workers. However, there are some problems where it is impracticable to use ideal functions because one has to compare the spaces corresponding to different problems.

A. Weinstein (College Park, Md.).

Michlin, S. G. Fundamental solutions of the dynamic equations of the theory of elasticity for non-homogeneous media. Akad. Nauk SSSR. Prikl. Mat. Meh. 11, 423–432 (1947). (Russian. English summary)

The object of this paper is to generalize the construction of singular displacement vectors, introduced by Volterra, to nonhomogeneous elastic media, and to obtain the generalized formula of Stokes giving the fundamental solution of the theory of elasticity for nonhomogeneous media. It is demonstrated that if the initial displacements and velocities of points in an infinite elastic medium are given, the generalized Stokes formula leads to a certain integro-differential equation for the displacements. *I. S. Sokolnikoff.*

Broch, Einar Klaumann. On the vibration spectrum of polar cubic crystal lattices. Arch. Math. Naturvid. 49, no. 2, 19–33 (1947).

Zuerst wird ein kurzer Ueberblick über Born's Theorie der freien Schwingungen eines Kristallgitters, welche zur Aufstellung der Frequenzgleichung führt, gegeben. Hierauf werden die Teilchen als geladen angenommen und es wird derjenige Teil der Koeffizienten in der Frequenzgleichung, der von den Coulombschen Wechselwirkungen herrührt, bestimmt, um das Schwingungsspektrum eines polaren Kristalls zu ermitteln. Das Problem wurde schon von Born und Thompson [Proc. Roy. Soc. London. Ser. A. 147, 594–599 (1934); 149, 487–505 (1935)] behandelt. Hier wird eine einfachere Methode vorgeschlagen, die von Born's ursprünglicher Frequenzgleichung ausgeht. In dieser allgemeinen Theorie sind die Kräfte zwischen den Teilchen allgemeine Zentralkräfte und die Koeffizienten in den Frequenzgleichungen haben die Form 3-dimensionaler Fourierreihen. Die Fourierreihen werden bei Annahme elektrostatischer Kräfte zwischen den Teilchen direkt mittels verallgemeinerter Zetafunktionen berechnet [vgl. Epstein, Math. Ann. 56, 615–644 (1903); 63, 205–216 (1907)]. Bei Beschränkung auf kubische Gitter wird der elektrostatische Teil der Koeffizienten in den Frequenzgleichungen berechnet.

W. Nowacki (Bern).

Gutman, S. G. A general solution of a problem of the theory of elasticity in generalized cylindrical coordinates. Doklady Akad. Nauk SSSR (N.S.) 58, 993–996 (1947). (Russian)

Cylindrical coordinates $(r, \theta, z)=(e^{-\alpha}, \beta, \gamma)$ and spherical coordinates $(r, \theta, \varphi)=(-1/\gamma, \beta, 2\tan^{-1}\alpha)$ are special instances of generalized cylindrical coordinates α, β, γ, for which

$$ds^2=\left(\frac{d\alpha}{h_\alpha}\right)^2+\left(\frac{d\beta}{h_\beta}\right)^2+\left(\frac{d\gamma}{h_\gamma}\right)^2,$$

$$h_\gamma=(a\gamma+b)^2,\quad h_\alpha=h_\beta=h_1h_\gamma^{\frac{1}{2}},\quad \frac{\partial h_1}{\partial\gamma}=0.$$

The components of strain in such coordinates are expressed in terms of a harmonic and biharmonic function and these expressions simplify considerably in the case of ordinary cylindrical and spherical coordinates. *T. C. Doyle.*

Mahover, E. V. Certain problems of the theory of plasticity of anisotropic media. Doklady Akad. Nauk SSSR (N.S.) 58, 209–212 (1947). (Russian)

The author applies R. von Mises' equations for the plastic deformation of crystals [Z. Angew. Math. Mech. 8, 161–185 (1928)] to the problems of torsion and plane strain. In the case of torsion, for instance, it is assumed that the crystal is twisted about a principal axis, that the torque is large enough to produce plastic flow throughout the crystal and that the velocity of the points in a generic cross section is obtained from a rotation of this cross section in its plane and a simultaneous warping which is the same for all cross sections. It is shown that this particular kinematic scheme is possible only in regular, hexagonal and monoclinic crystals. Plane strain is discussed along similar lines. Here, too, it is found that the basic equations are satisfied only in the regular, hexagonal and monoclinic systems.

W. Prager (Providence, R. I.).

Sokolovskiĭ, V. V. Certain problems of the statics of plastic and granular substances. Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1947**, no. 10, 1275–1286 (1947). (Russian)

A brief survey of the author's recent contributions to the theory of plastic or granular media [cf. the author's books "Statics of Earthy Media," 1942; "Theory of Plasticity," 1946; these Rev. **6**, 27; **8**, 545]. *W. Prager.*

Neronov, N. P. Determination of the strains in a lifting cable. Doklady Akad. Nauk SSSR (N.S.) **57**, 765–768 (1947). (Russian)

An elastic cable, wound by a drum rotating about a fixed horizontal axis, lifts a weight. The strain is studied subject to the boundary conditions that the strain vanishes at the weighted end of the cable and that the cable passes over the drum without slip. Initial conditions assume the drum to wind the cable with constant rim acceleration prior to the instant of observation, when the rim acceleration becomes and remains zero. Boundary and initial conditions reduce the partial differential equation for the strain to a third order ordinary equation expressing the derivatives for larger values of the independent variable in terms of derivatives for smaller values. The tension at the junction of the cable with the weight for any subsequent time is found by a method of successive integrations which progressively extend the interval of the independent variable as the weight rises toward the drum. *T. C. Doyle* (Hanover, N. H.).

Rellich, F. Die Randbedingungen der Airyschen Spannungsfunktion bei vorgegebenen Randverschiebungen. Z. Angew. Math. Mech. **25/27**, 13–17 (1947).

This note is concerned with the formulation of the boundary conditions in the theory of finite bending of thin plates. It is shown that the conditions of prescribed edge displacements u, v, w can be expressed in terms of w and in terms of Airy's stress function Φ. The form of these conditions is obtained. The results include the special case of generalized plane stress for which $w=0$. *E. Reissner.*

Odley, Ezra G. Deflections and moments of a rectangular plate clamped on all edges and under hydrostatic pressure. J. Appl. Mech. **14**, A-289–A-299 (1947).

The deflections considered are small in comparison with the thickness of the plate. The differential equation satisfied by the deflection w is $\Delta^4 w=$constant. The boundary conditions are $w=dw/dn=0$, where n denotes the outer normal. An approximate solution is obtained in the same manner as in Timoshenko's book [Theory of Plates and Shells, McGraw-Hill, New York, 1940, pp. 222–232]. A second approximate solution is then obtained by the method of H. Marcus [Bauingenieur **17**, 40–44 (1936); Ann. Ponts Chaussées **107 I**, 538–567 (1937)]. In this method, a solution is sought in the form $w=X(x)Y(y)$. First X is guessed, and a corresponding first approximation Y_1 to Y is determined by making the total load on each strip parallel to the y-axis have the correct value. Then, by using Y_1, a corresponding first approximation X_1 to X is determined by making the total load on each strip parallel to the x-axis have the correct value. This process is repeated until the desired accuracy is attained. Numerical results are obtained for the square plate, for both solutions. The second solution yields larger values, but the difference is small. *G. E. Hay* (Ann Arbor, Mich.).

Ilyushin, A. A. The elasto-plastic stability of plates. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **1188**, 30 pp. (1947).

Translation of Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **10**, 623–638 (1946); these Rev. **8**, 359.

Ling, Chih-Bing. Stresses in a notched strip under tension. J. Appl. Mech. **14**, A-275–A-280 (1947).

An infinitely long strip having uniform width except for a pair of semi-circular notches is subject to longitudinal tension. The stress function is found as a superposition of that appropriate for the uniform strip and certain biharmonic functions which are singular at the notch centers. These latter functions are found as the derivatives of certain well-known solutions of classical elasticity problems. The results agree very well with experimental evidence. *G. F. Carrier* (Providence, R. I.).

Wang, Tsun Kuei. Buckling of transverse stiffened plates under shear. J. Appl. Mech. **14**, A-269–A-274 (1947).

Marčenko, V. The momentless spherical shell for large displacements. Doklady Akad. Nauk SSSR (N.S.) **57**, 21–24 (1947). (Russian)

The author considers a closed spherical shell subjected to a uniform internal pressure and compressed between two parallel planes. Equations of momentless shell theory are used and the displacements are supposed to be large. Using Gauss-Codazzi relations, Hooke's law and the equilibrium equations, the author obtains a system of equations describing the state of stress in an arbitrary axially symmetric shell. *I. S. Sokolnikoff* (Los Angeles, Calif.).

Colin, E. C., Jr., and Newmark, N. M. A numerical solution for the torsion of hollow sections. J. Appl. Mech. **14**, A-313–A-315 (1947).

Klitchieff, J. M. Torsion of a rectangular tube. J. Appl. Mech. **14**, A-287–A-288 (1947).

Hadji-Argyris, J., and Dunne, P. C. The general theory of cylindrical and conical tubes under torsion and bending loads. Single and many cell tubes of arbitrary cross-section with rigid diaphragms. III. J. Roy. Aeronaut. Soc. **51**, 884–930 (1947).

For the first two parts cf. the same vol., 199–269, 757–784 (1947); these Rev. **8**, 613; **9**, 122.

MacGregor, C. W., and Coffin, L. F., Jr. Approximate solutions for symmetrically loaded thick-walled cylinders. J. Appl. Mech. **14**, A-301–A-311 (1947).

Ghosh, S. On the flexure of an isotropic elastic cylinder. Bull. Calcutta Math. Soc. **39**, 1–14 (1947).

The paper contains a discussion of flexure of an isotropic elastic cylinder drawn from secs. 52, 53, 54 and 58 of I. S. Sokolnikoff's Mathematical Theory of Elasticity [McGraw-Hill, New York, 1946]. *I. S. Sokolnikoff.*

Vasilesco, Florin. Sur le flambement des poutres droites à section constante et à moment d'inertie variable. C. R. Acad. Sci. Paris **225**, 716–718 (1947).

The buckling of beams of variable bending stiffness is investigated. The integral equation is written down and formulae for the eigenvalues are given. The method of solution is not stated but it appears that the conventional iteration procedures for the Fredholm equation solution form the basis for the results. *G. F. Carrier.*

MATHEMATICAL PHYSICS

Optics, Electromagnetic Theory

Marton, L., and Bol, K. Spherical aberration of compound magnetic lenses. J. Appl. Phys. **18**, 522–529 (1947).

The purpose of this paper is to reduce the error due to spherical aberration for an electron optical system, by introducing additional lenses. The authors in the first place give a review of Glaser's magnetic lens [see Glaser and Lammel, Arch. Elektrotechnik **37**, 347–356 (1943); these Rev. **7**, 398]. After a brief exposition they consider the case of two lenses [see Glaser, Z. Physik **117**, 285–315 (1941); these Rev. **4**, 32]. The secondary lens is taken to be a weak lens whose purpose is to bring the virtual image due to the first lens into a real image. The fields here are such that there is no interaction between them. The error is found to be a linear combination of each individual lens as derived from Glaser's treatment. The result is that for nonoverlapping fields the only way to decrease the spherical aberration is to decrease the distance of the image from the center of the first (strong) lens. This, however, is not satisfactory, for the magnification of the lens approaches unity, an obvious disadvantage. Another possibility is to decrease the ratio of the half widths of the lenses, but this again leads to an increase of the distance of the image in the same ratio. Still another alternative is to decrease the refractive power of the second lens (K_2). However, this will produce a large focal length which requires a large magnification of the first lens in order to produce a virtual image beyond the focal point of the second lens. But a large magnification of the first lens gives a large error; hence there is an optimum value for K_2 which will result in giving a real image. The results of the analysis are plotted graphically. Finally a three-lens system is considered. The authors give an expression for the coefficient of aberration for two weak lenses combined with a strong lens. They conclude that for such a system the aberration is smaller than for the two-lens system provided that the half width of the second lens is much smaller than that of the third. The image distance is found to depend on that of the last lens. The result of this analysis is shown in several graphs. *N. Chako.*

Hammad, A. The primary and secondary scattering of sunlight in a plane-stratified atmosphere of uniform composition. III. Numerical tables and discussion of secondary scattered light. Philos. Mag. (7) **38**, 515–529 (1947).

Numerical results of the theory developed in previous papers [Philos. Mag. (7) **28**, 99–110 (1939); **36**, 434–440 (1945); these Rev. **7**, 270] are presented. The directional distribution of the secondary scattered light and its deviation from isotropy are discussed. In general the radiation scattered more than once is found to be approximately isotropic. *W. E. K. Middleton* (Ottawa, Ont.).

Smythe, W. R. The double current sheet in diffraction. Physical Rev. (2) **72**, 1066–1070 (1947).

The electromagnetic radiation from an opening in a thin perfectly conducting plane screen is shown to be identical with that produced by a double current sheet fitting the opening. The current density, being equal and opposite in the two layers of the sheet, is proportional to the initial tangential electric field over the opening. The theory is applied to a small circular hole [H. A. Bethe, same Rev. (2) **66**, 163–182 (1944); E. T. Copson, Proc. Roy. Soc. London. Ser. A. **186**, 100–118 (1946); these Rev. **6**, 165; **8**, 179]. Bethe's result is derived by means of spheroidal potential functions. A second application concerns the diffraction from a rectangular aperture; it immediately leads to known results of Stratton and Chu [same Rev. (2) **56**, 99–107 (1939)]. Finally, the problem of the excitation of a rectangular wave guide by a coaxial line termination in one side is solved for the TE_{10}-mode. An expression for the corresponding output resistance is given. *C. J. Bouwkamp* (Eindhoven).

Abelès, Florin. Sur la réflexion et la transmission d'une onde plane par une lame absorbante. C. R. Acad. Sci. Paris **225**, 1297–1298 (1947).

Blaisse, B. S., and van der Sande, J. J. On the exact calculation of the reflectance of glass, coated with an arbitrary number of non-absorbing layers. Physica **13**, 413–416 (1947).

da Silveira, A. On the fundamental vector of the magnetic field and the magnetostatic pressure. Philos. Mag. (7) **38**, 339–342 (1947).

The author investigates the properties of the vectors, magnetic induction **B** and magnetic intensity **R**, due to a surface distribution of currents. It is shown that the normal component of **B** is continuous, and that $[\mathbf{n}, \mathbf{R}-\mathbf{B}^{\pm}] = \mp 2\pi\mu_0 c^{-1}\mathbf{j}$ (μ_0 and $\mathbf{j}$ are the permeability and current density, respectively), which upon subtraction lead to the well-known result $[\mathbf{n}, \mathbf{B}^{+}-\mathbf{B}^{-}] = 4\pi\mu_0 c^{-1}\mathbf{j}$. It is also shown that the magnetostatic pressure is directed towards the interior of the conductor. *C. Kikuchi* (East Lansing, Mich.).

Davy, N. A closer approximation to the theory of the field of Gerlach-Stern magnets. Philos. Mag. (7) **36**, 852–859 (1945).

The author solves the following two-dimensional potential problem. A semi-infinite thin plate at potential $V = V_1$ faces a slotted infinite plate kept at zero potential. The depth of the slot is f, the width is $2l$. The first plate (AB) intersects the plane of z along $-\infty < z < -h$; OB represents a line of force; the slotted plate is represented by the segments ON, NM, ML, and their reflections in the real z-axis, where at O, N, M, L, $z = 0, il, -f+il, -f+i\infty$, respectively. The interior of the polygon $ABONML$ is mapped onto the upper half of the t-plane ($t = -\infty, -\alpha, 0, k^2, 1, \infty$ corresponding, in order, to $A, \cdots, L$), in terms of a parameter u defined by $1/t = \mathrm{sn}^2(u, k)$, viz.,

$$z = (il)/E \cdot \{uE/K + Z(u+iK') + i\pi/2K\} - f,$$

where k, the modulus of the Jacobian elliptic functions, is determined by the dimensions of the slot, viz., $f/l = (K'-E')/E$. (Here $u = -iK'$ at $z = 0$, not $+iK'$.)

To link the geometrical conditions with the electrical, a rectangle $A'B'O'L'$ ($t = -\infty, -\alpha, 0, \infty$) in the plane of $W = U + iV$ is also transformed into the upper half of the t-plane. This transformation is accomplished by elementary functions (in t), leading to

$$W = 2V_1\pi^{-1} \log \left\{ \frac{\mathrm{sn}\,\beta + (\mathrm{sn}^2\beta + \mathrm{sn}^2 u)^{\frac{1}{2}}}{\mathrm{sn}\,u} \right\},$$

where β depends on h; $A'B'$ is at potential V_1, $O'L'$ at potential zero. The equipotentials and the lines of force are easily

found in the t-representation. They "do not seem obtainable by theory" for the z-representation. A numerical example is worked out; some quantities of physical importance are discussed. *C. J. Bouwkamp* (Eindhoven).

Pekeris, C. L. The field of a microwave dipole antenna in the vicinity of the horizon. II. J. Appl. Phys. **18**, 1025–1027 (1947).

The results of a previous paper by the same author [same J. **18**, 667–680 (1947); these Rev. **9**, 126] are extended in a natural way. *C. J. Bouwkamp* (Eindhoven).

Müller-Strobel, J., und Patry, J. Berechnung des Stromes der Rahmenempfangsantenne. Schweiz. Arch. Angew. Wiss. Tech. **13**, 193–202 (1947).

The author uses Hallén's integral equation method to obtain a first order approximation to the current at the base of a receiving loop antenna. It is assumed that the loop is circular, of radius $a \ll \lambda$, and has N turns of thin wire of radius $\rho \ll a$. *M. C. Gray* (New York, N. Y.).

Bouwkamp, C. J. Calculation of the input impedance of a special antenna. Philips Research Rep. **2**, 228–240 (1947).

The author obtains formulas for the input impedance of a vertical antenna fed against a system of two or four horizontal wires, assuming that all the wires are of the same dimensions and that the current distribution is sinusoidal throughout. For a quarter-wave antenna the relation $Z_0+6Z_2=8Z_4$ is derived, where Z_0 is the input impedance of the antenna fed against an infinite perfectly conducting plate, and Z_2 and Z_4 the corresponding impedances for the two and four wire systems. *M. C. Gray.*

Ingram, W. H. Note on the integral equation of the electrical transmission line. Philos. Mag. (7) **38**, 61–64 (1947).

The author uses the example of the integral equation of the electrical transmission line as a physical problem involving a "proper" Green's matrix, i.e., a matrix of one-dimensional Green's functions in which at least two elements do not vanish. A detailed example is given. *A. E. Heins* (Pittsburgh, Pa.).

Graffi, Dario. La teoria dei circuiti elettrici e le equazioni di Maxwell. Rend. Circ. Mat. Palermo **62**, 249–285 (1940).

***Courant, Ernest D. Current distribution in an ironless synchrotron magnetic field.** Studies and Essays Presented to R. Courant on his 60th Birthday, January 8, 1948, pp. 85–94. Interscience Publishers, Inc., New York, 1948. \$5.50.

In order to produce a magnetic field distribution within the active toroidal space of a synchrotron leading to stable particle orbits, the author examines the required density variation of a current sheath just surrounding the elliptical toroid of magnetic field. By use of a Fourier cosine expansion and contour integration of the component coefficients an approximate solution is achieved for the first few terms. Some consideration is also given to the mechanical forces exerted on the current sheath and the magnetic field energy required. *E. Weber* (Brooklyn, N. Y.).

Livens, G. H. On magnetic field theories. Philos. Mag. (7) **38**, 453–479 (1947).

Elsasser, Walter M. Induction effects in terrestrial magnetism. III. Electric modes. Physical Rev. (2) **72**, 821–833 (1947).

The analysis given in two earlier papers [same Rev. (2) **69**, 106–116 (1946); **70**, 202–212 (1946); these Rev. **7**, 401; **8**, 186] is extended by considering the set of solutions of the field equations which give rise to modes of the electric type. In particular, the effect of the quadrupole term on the earth's magnetic field is examined. It is concluded that the field due to this term is likely to be much larger than the ordinary magnetic dipole field and that a possible source of power needed to maintain this mode is the rotational energy of the earth. *C. Kikuchi* (East Lansing, Mich.).

Tihonov, A. N. On electric sounding above a sloping layer. Akad. Nauk SSSR. Trudy Inst. Teoret. Geofiz. **1**, 116–136 (1946). (Russian)

A problem of electrical prospecting by DC is studied in this paper. Given a point source on the horizontal plane (ground) $z=0$ above a sloping and outcropping layer L of infinite thickness the distribution of potential in this plane $z=0$ is required provided that the specific resistance ρ_1 (resistivity) of the layer L is different from that ρ_2 of the other part of the half-space $z\leqq 0$ (vertical axis OZ being directed upwards). The problem is reduced to an integral equation which is solved by the method of successive approximations. The solution is transformed into a form well adapted to practical computations of the observed apparent resistivity. *E. Kogbetliantz* (New York, N. Y.).

Tyurkišer, R. I. Calculation of the field of a point source in the presence of a sloping layer. Akad. Nauk SSSR. Trudy Inst. Teoret. Geofiz. **1**, 137–142 (1946). (Russian)

Performing an integration involved in the paper reviewed above; routine work with elliptic integrals. *E. Kogbetliantz* (New York, N. Y.).

Quantum Mechanics

Fock, V., and Krylov, N. On the uncertainty relation between time and energy. Acad. Sci. USSR. J. Phys. **11**, 112–120 (1947).

Two distinct types of uncertainty relation between time and energy in quantum mechanics are distinguished. Though expressed by similar formulas, such as $\Delta H \Delta T \geqq h/2$, they have different interpretations. The Bohr type connects the change in energy of a single particle, during a measurement, with the duration of the measurement. The Mandelstam-Tamm type [same J. **9**, 249–254 (1945); these Rev. **7**, 404] connects the standard deviation of the energy of a statistical aggregate, such as a wave packet, with the least time during which the mean value of an observable can change by an amount equal to its own standard deviation. The Schrödinger equation holds in the latter case but not the former. The Mandelstam-Tamm relation is then applied to the question of the connection between the energy distribution function of a state and the law of decay of the state. *O. Frink* (State College, Pa.).

de Broglie, Louis. Sur la fréquence et la vitesse de phase des ondes planes monochromatiques en mécanique ondulatoire. C. R. Acad. Sci. Paris **225**, 361–363 (1947).

Of the four quantities wave length λ, group velocity U, phase velocity V and frequency ν, only the first two are

directly measurable for a plane monochromatic wave in wave mechanics. The relations $V=\lambda\nu$ and $U=d\nu/d\mu$ (where $\mu=1/\lambda$) determine the last two quantities in terms of the first two and an additive constant ν_0. The author shows that, if relativistic invariance is assumed, the constant ν_0 can be determined. He thus derives the explicit formula $\nu(\mu)=\int_0^\mu U(\mu)d\mu/(1-(1-U^2/c^2)^{\frac{1}{2}})$ which together with the relation $V=\lambda\nu$ serves to express V and ν in terms of U and λ. This formula lacks the Planck constant h, since it involves only quantities related to the wave aspect, as opposed to the corpuscular aspect. *O. Frink.*

Yvon, Jacques. La notion quantique de mélange et ses applications. J. Phys. Radium (8) 8, 182–184 (1947).

Jastrow, Robert. On the Rydberg-Ritz formula in quantum mechanics. Physical Rev. (2) **73**, 60–67 (1948).

A derivation is given of the Rydberg-Ritz formula for series spectra. Simple expressions are obtained for the Rydberg and Ritz coefficients. It is shown that the Ritz coefficient is proportional to the difference between the radial period of the electron and the period of the hypothetical orbit, with the same energy which would exist if the atomic core were to contract to zero radius. The derivation necessitates a study of the confluent hypergeometric function. An expansion in powers of the energy is obtained for this function. The coefficients in the expansion are found to be simple combinations of Bessel functions.

From the author's summary.

Yukawa, H. Modern physics and mathematics. Tensor 7, 6–15 (1944). (Japanese)

This is an address at a meeting of the Tensor Society. First the author makes clear the essential difference between theoretical physics and mathematics and also their close connection by taking quantum mechanics as an instance. Next he states the necessity of introducing the theory of relativity into the theory of elementary particles. As we must there analyse "nuclear field" mathematically, he hopes for the appearance of a suitable mathematics for this purpose. *A. Kawaguchi* (Sapporo).

Vrkljan, V. S. Zur Ableitung des magnetischen Moments des Elektrons. Proc. Indian Acad. Sci., Sect. A. **25**, 523–528 (1947).

The author shows that with a proper choice of the first three Dirac matrices α_1, α_2, and α_3, the fourth Dirac matrix α_4 may be chosen in infinitely many ways so that, on the one hand the correct commutation rules hold, and on the other hand the usual formulas for the current density vector and the magnetic moment of the electron are obtained, using the wave-packet method of C. G. Darwin.

O. Frink (State College, Pa.).

Eliezer, C. Jayaratnam. Quantum electrodynamics and low-energy photons. Proc. Roy. Soc. London. Ser. A. **191**, 133–136 (1947).

In a previous paper by the author [same Proc. Ser. A. **187**, 210–219 (1946); these Rev. **8**, 123] a scattering process involving three photons was discussed using Dirac's formulation of quantum electrodynamics and the expansion of wave functions in powers of e^2/hc. It was shown then that the probability of multiple emission of photons in a scattering process would be large provided some of the emitted photons are of small energy. In this paper the author integrates the probability of the incident photon's dividing into two photons, with respect to the frequency of one of them, and shows that the result is infinite. He concludes that in dealing with processes involving low energy photons one has for the present to give up the expansion in powers of e^2/hc. *A. H. Taub* (Princeton, N. J.).

Costa de Beauregard, Olivier. Sur la symétrisation relativiste du formalisme quantique en théorie de Dirac. C. R. Acad. Sci. Paris **225**, 626–629 (1947).

The author generalizes the definition of scalar product of two four-component spinor functions usually used in the Dirac theory of the electron and evaluates the expectation values of some operators. The generalization consists of replacing the integration over the hyperplanes in which time is constant by integration over members of a family of hypersurfaces whose normals are always time-like.

A. H. Taub (Princeton, N. J.).

Becker, R. Die aus der Dirac-Gleichung des Elektrons folgende Zwei-Komponenten-Gleichung. Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. Math.-Phys.-Chem. Abt. **1945**, 20–28 (1945).

This paper is concerned with the study of the nonrelativistic approximation of the Dirac equations for an electron in an external electromagnetic field. A scheme is given for carrying out the approximation so as to include terms involving any negative power of $\mu=2mc$, where m is the mass of the electron and c is the velocity of light. The scheme involves the replacement of the four-component spinors of the Dirac equation by two-component spinors by using a contact transformation containing powers of $1/\mu$. The detailed calculation is carried out in which terms in $1/\mu^3$ are retained. It is then shown that the Hamiltonian is then the usual nonrelativistic one with spin interactions plus an additional term $\{h^2e/(8m^2c^2)\}\operatorname{div} E$. In the previous treatments of this problem the nonrelativistic approximation was not obtained by a contact transformation and imaginary terms appeared in the nonrelativistic Hamiltonian. The relation between these imaginary terms and the term given above is discussed. However, it is not pointed out that the term given above is a manifestation of the high frequency oscillation of the electron about its mean position due to the negative energy states. *A. H. Taub.*

Peng, H. W. On the representation of the wave function of a quantized field by means of a generating function. Proc. Roy. Irish Acad. Sect. A. **51**, 113–122 (1947).

It is shown that any wave function describing an assembly of bosons and fermons may be obtained from the wave function corresponding to no particles by means of a generating quantity, which is a q-number, so that there is a one-to-one correspondence between the set of wave functions and the set of generating quantities. It is then found that when wave functions are represented by their generating quantities F the basic q-numbers ξ, ξ^* (bosons), η, η^* (fermons) are representable by multiplication and partial derivative operators: $\xi\to\xi$, $\xi^*\to\partial/\partial\xi$; $\eta\to\eta$, $\eta^*\to\partial/\partial\eta$. These derivatives are of course with respect to q-numbers and are defined by first putting F into a well-ordered form. The wave equation for the quantized field is then

$$ih\partial F/\partial t=HF \quad (H=H(\xi,\partial/\partial\xi,\eta,\partial/\partial\eta)),$$

which is a normal partial differential equation except that the η's anticommute. This latter difficulty may be overcome by the introduction of complex variables and a generating function Φ of these variables which stands in one-to-one

correspondence with F. The η, η^* may then be represented by integral operators and the equation for Φ may be treated by ordinary analysis. As an example of the use of Φ it is shown that the question as to whether a given Hamiltonian admits normalizable solutions of the wave equation may be determined from the asymptotic form of Φ.

H. C. Corben (Pittsburgh, Pa.).

Ma, S. T. Relativistic invariance of the quantum theory of radiation. Physical Rev. (2) **72**, 1090–1096 (1947).

It is shown that the current formulation of the quantum theory of radiation, which is concerned solely with the quantization of the transverse part of the electromagnetic field, is in conformity with the principle of relativity. The relativistic invariance of the quantum theory of the neutral-vector meson field is also discussed. *Author's summary.*

Heitler, W., and Hu, N. Proton isobars in the theory of radiation damping. Proc. Roy. Irish Acad. Sect. A. **51**, 123–140 (1947).

A desideratum of Heitler's radiation damping theory has been that since it allows one to calculate only the asymptotic forms of wave functions for a system it would appear that it could not be applied to the problem of calculating energy eigenvalues of bound states of mesons and nucleons or to other static problems. It is now pointed out that it is possible to perform such calculations according to the radiation damping theory. This is done most conveniently by fitting the theory into the scheme of the Heisenberg S-matrix for which it provides a possible form. Diagonalizing S, one looks for solutions of $S=0$ which correspond to negative imaginary values of the momenta. These correspond to energy eigenvalues of bound states. Similarly metastable states are given by solutions for which k is complex, the boundary condition at $r=\infty$ again requiring that the imaginary part be negative.

Isobaric states are found for a single meson bound to a nucleon for the cases of longitudinal mesons and of the Møller-Rosenfeld mixture. The energies of these states are estimated but it is pointed out that slight changes in S may have a large effect on the results without altering appreciably the scattering cross-sections.

H. C. Corben (Pittsburgh, Pa.).

Lattes, C. M., Schönberg, M., and Schützer, Walter. Classical theory of charged point-particles with dipole moments. Anais Acad. Brasil. Ci. **19**, 193–245 (1947).

Extension of Schönberg's classical theory of the point electron [Summa Brasil. Math. **1**, 41–75, 77–114 (1946); Physical Rev. (2) **69**, 211–224 (1946); these Rev. **8**, 427, 428] to a charged point particle with an intrinsic magnetic dipole moment. As in the papers cited, the fields are divided into "attached" and "radiated" parts. The (asymmetrical) energy momentum tensor is derived. The Hamiltonian formalism is extended to the particles under consideration. There occurs a coupling between spin rotation and translational motion. *A. Pais* (Princeton, N. J.).

Petiau, Gérard. Sur la réflexion des corpuscules de spin demi-entier. Revue Sci. **85**, 135–142 (1947).

An analysis of the so-called Klein paradox for particles of spin $(2p+1)h/4\pi$, p a whole number. One considers the solutions of the wave equations for the particles concerned in the presence of a potential $U=0$, $x<0$, $U=$a constant $U_0>0$ for $x>0$. The plane wave solutions involve $p+1$ possible values for the rest mass of the particle. If a plane wave, corresponding to one given rest mass, is incident on the potential U_0 it is possible (if $p>0$) that the reflected as well as the transmitted wave has components pertaining to rest masses differing from the original one; U_0 thus acts as a "mass filter." *A. Pais* (Princeton, N. J.).

Petiau, Gérard. Les équations d'ondes du second ordre dans la théorie du méson. J. Phys. Radium (8) **8**, 116–122 (1947).

A discussion of the equations for meson fields of spin 0 and 1 in interaction with the electromagnetic field. From the first order differential equations for the meson field strengths, second order equations are derived by iteration. These second order equations involve terms describing the magnetic moment of the meson. The problem is discussed from the point of view of both the "field aspect" and the "particle aspect" of meson theory, while the de Broglie formulation is also considered. *A. Pais.*

Kwal, Bernard. Sur le potentiel et la transformation de jauge en théorie du corpuscule de spin 1. C. R. Acad. Sci. Paris **225**, 922–923 (1947).

Kwal, Bernard. Approximation de l'optique géométrique en théorie du corpuscule de spin 1. C. R. Acad. Sci. Paris **226**, 61–63 (1948).

Potier, Robert. Sur la représentation d'un corpuscule de spin 1 à masses multiples. C. R. Acad. Sci. Paris **226**, 63–64 (1948).

BIBLIOGRAPHICAL NOTES

★Den 10. Skandinaviske Matematiker Kongres i København 26.–30. August 1946. Comptes Rendus du Dixième Congrès des Mathématiciens Scandinaves Tenu à Copenhague 26.–30. Août 1946. Jul. Gjellerups Forlag, København, 1947. xxv+383 pp. 25 Danish Kroner, bound.

In addition to the proceedings of the Congress, this volume contains forty-one lectures, which have been reviewed separately in Mathematical Reviews.

Communications on Applied Mathematics.

The first issue of this quarterly is dated January, 1948. It is issued by the New York University Institute for Mathematics and Mechanics.

★Studies and Essays Presented to R. Courant on his 60th Birthday, January 8, 1948. Interscience Publishers, Inc., New York, 1948. viii+470 pp. $5.50.

The thirty-eight papers contained in this volume are being reviewed separately in Mathematical Reviews.

Monatshefte für Mathematik.

This continues Monatshefte für Mathematik und Physik. The first issue under the new title is vol. 52, no. 1, issued in March, 1948.